RESEARCH CENTER

COLLEGE OF ENGINEERING

UNIVERSITY OF PUERTO RICO

MAYAGUEZ CAMPUS

Joint Flexibility Effects on the Dynamic Response of Structures - Part II : Stochastic Analysis

Final Report

by

Luis E. Suarez and Enrique E. Matheu

December 1992

prepared for

U.S. Army Research Office

Grant Number 28374-EG-H

University of Puerto Rico - Mayagüez

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
December 1992

3. REPORT TYPE AND DATES COVERED
Final Report

4. TITLE AND SUBTITLE
Joint Flexibility Effects on the Dynamic Response of Structures - Part II: Stochastic Analysis

5. FUNDING NUMBERS
DAAL03-90-G-0214

6. AUTHOR(S)
Luis E. Suarez and Enrique E. Matheu

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
University of Puerto Rico - Mayaguez
College of Engineering
P.O. Box 5000
Mayaguez, P.R. 00681-5000

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
ARO 28374.6-EG-H

11. SUPPLEMENTARY NOTES
The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

The objective of this study is to quantify the effect of flexible connections with random stiffness on the modal properties and dynamic response of structures. Since the mass and stifness matrices contain random parameters, the modal decomposition solution of the equations of motion lead to a random eigenvalue problem. A second order perturbation technique is used for the solution the random eigenproblem. Closed form expressions for the mean and variance of the natural frequencies and vibration modes are provided. The response of systems with random parameters subjected to deterministic loads is examined next beginning with single degree of freedom systems. When standard perturbation methods are applied to calculate the statistics of the response, the expressions obtained become non uniform as time grows. A detailed discussion about the cause of the problem is presented. A new formulation based on the method of multiple scales used for nonlinear systems is proposed to overcome the problem. The response analysis is extended to multi degree of freedom systems. Conclusions and reommendations for further research are presented.

14. SUBJECT TERMS
Flexible Connections - Random Eigenvalue Problems - Perturbation Techniques - Stochastic Finite Elements

15. NUMBER OF PAGES
64

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

DISCLAIMER

The view, opinions, and/or findings contained in this report are those of the authors and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

SCIENTIFIC PERSONNEL

The study presented in this report was carried out under the direction of the principal investigator, Dr. Luis E. Suarez, associate professor in the Department of General Engineering at the University of Puerto Rico-Mayaguez. He was assisted by Mr. Enrique E. Matheu, graduate student in the Department of Civil Engineering. Mr. Matheu completed his Master of Science degree in June 1992. He is now pursuing a Ph.D. degree in Engineering Mechanics at Virginia Polytechnic Institute. An undergraduate student, Mr. Jose Perez Suarez, also participated in the project. Mr. Perez completed his B.Sc. degree in June 1992 and then enrolled in the Master of Science in Civil Engineering program in UPR-Mayaguez.

Contents

List of Figures

List of Tables

Chapter 1

The Random Structural Eigenvalue Problem

1.1 Introduction

In this chapter we will take the first step in the direction of incorporating in the dynamic analysis of the structures the uncertainties associated with the values of the connection stiffnesses.

Under the assumption that the moment-rotation relationship for the joints is linear, each connection stiffness is represented by a single parameter, i.e., the slope of the $M - \theta$ curve. This parameter should be determined from a series of tests performed for each type of connection and for a given size of the structural elements that converge to the joint. The results presented in [1], [2] and [3] may be mentioned as examples of such experimental investigations. In particular, Altman et al. [3] identified the most important parameters that control the initial stiffness in bolted beam-column connections. Any change in the value of these parameters will cause a variation in the linear relationship assumed.

Although using the test results it is possible to assign a value to the rotational stiffness of a given connection configuration, one cannot guarantee that this value will be the same for all the connections of the same type in a given structure. The values of the connection stiffness obtained from a test should be regarded as the mean value of the stiffness of connections with the same configuration. There will always be a degree of dispersion due to the variations in the several parameters that control the connection behavior. Therefore, it is logical to consider the connection stiffness as a random variable in the structural model. In this case both the stiffness and the mass matrix will have random coefficients, since both are function of the fixity factors, which in turn are defined in terms of the connection stiffness. The finite element equations of motion of the structure become a set of ordinary differential equations with time-independent random coefficients. Although the equations of motion are random they are still linear and hence they could, at least in principle, be decoupled and solved by modal analysis. The eigenvalue problem associated with the equations of motion also becomes random, and the eigenvalues and eigenvectors of the system end up being random variables as well. The object of this chapter is to study this random algebraic eigenvalue problem associated with a structural system with random flexible joints. The dynamic response of these systems will be addressed in the following chapter.

1.2 Connection Stiffness as a Random Variable

The determination of the probability associated with a particular value of stiffness requires the knowledge of either its distribution function or its density function. In general, the choice of a distribution to describe the probabilistic behavior of a physical variable is governed by the nature of the phenomenon. In the particular case of a variable that represents the total effect of several random causes, the central limit theorem leads to the conclusion that its distribution is asymptotically normally distributed. As it was mentioned before, there are a number of factors that contribute to the variability of the stiffness of a given connection. Consequently, it is reasonable to assume that the stiffness values can be modelled as random variables with normal or Gaussian distribution.

In our case, the value of the connection stiffness can be regarded as the sum of a deterministic component and a component representing random perturbations due to the variability in the controlling parameters. These variations can be caused, for example, by fluctuations in the material characteristics, the geometry of the connection, the assembling method or the quality control of the process. Therefore, the stiffness value of the $i-th$ structural joint can be represented as follows:

$$K_i = \bar{K}_i + \alpha_i \tag{1.1}$$

where K_i is the random rotational stiffness of the joint, $\bar{K}_i$ is a deterministic constant and α_i is a random perturbation. In order to simplify the analysis the mean value of the random perturbation will be incorporated into the deterministic component so that α_i can be regarded as a zero-mean random variable. Moreover, if we assume that the probability distribution of the connection stiffness is normal, then we conclude that α_i is also normal, since normal random variables remain normal under linear transformations. With the above considerations it is straightforward to show that the first two moments of the random variables α_i and K_i are given by:

$$E\{\alpha_i\} = 0 \quad ; \quad E\{K_i\} = \bar{K}_i \tag{1.2}$$

$$E\left\{(\alpha_i)^2\right\} = \sigma^2_{\alpha_i} = \sigma^2_{K_i} \quad ; \quad E\left\{(K_i)^2\right\} = \sigma^2_{K_i} + \bar{K}_i^2 \tag{1.3}$$

Hence, the mean value of the stiffness is $\bar{K}_i$, and its variance $\sigma^2{}_{K_i}$ is equal to the variance $\sigma^2_{\alpha i}$ of the random perturbation α_i . These two quantities define completely the probability distribution.

1.3 Second Order Perturbation Technique

In the formulation of a probabilistic finite element method based on the second order perturbation technique, each random variable is expanded about its mean value and terms up to second order are retained. The rates of change of the eigenproperties with respect to the fixity factors calculated in the previous chapter will be used to obtain expressions for the mean and variances of the eigenproperties in terms of the first and second moments of the random stiffnesses. For this reason, this type of approach is called a *second − moment* analysis. The inherent limitation

of this formulation is that the statistical variations of the random variables have to be small in order to obtain acceptable accuracy, as it is the case with all perturbation-based methods.

If the random parameters of the structure are substituted by their means or expected values, we obtain an averaged version of the structural system. Therefore, using the mean values of the joint stiffness to define the system matrices, the solution of the associated eigenvalue problem results in a deterministic set of eigenproperties. It will be shown later that these eigenproperties coincide with the mean values of those obtained through a perturbation method based on a first order expansion. Hence, to study the difference between the deterministic eigenproperties of the averaged system and the mean values of the eigenvalues and eigenvectors associated to the random system, it is necessary to include at least second order terms in the expansions.

The eigenvalue problem for a structural system with N unconstrained degrees of freedom and R random flexible joints is:

$$\{[K(\boldsymbol{K})] - \lambda_i(\boldsymbol{K})\,[M(\boldsymbol{K})]\}\,\boldsymbol{\phi}_i(\boldsymbol{K}) = \mathbf{0} \quad ; \quad i = 1,2,...,N \tag{1.4}$$

where $[K]$ and $[M]$ are the stiffness and mass matrices whose coefficients are function of the random variables $K_1, K_2, ..., K_R$. These variables are expressed in vector form as follows:

$$\boldsymbol{K} = \{K_1\, K_2 \ldots K_R\}^T \tag{1.5}$$

The eigenvalues λ_i and eigenvectors $\boldsymbol{\phi}_i$ become nonlinear functions of the variables $K_1, K_2, ..., K_R$ and hence they are also random quantities. The specific form of the nonlinear functions cannot be determined, except for trivial cases. Nevertheless, assuming that the variables K_i are constrained to small fluctuations about their mean values, we can express the eigenproperties as Taylor series expansions in terms of the random stiffness parameters. The coefficients of the expansions are evaluated at the mean value vector defined as follows:

$$\bar{\boldsymbol{K}} = \{\bar{K}_1\, \bar{K}_2 \ldots \bar{K}_R\}^T \tag{1.6}$$

Therefore, the eigenvalues can be approximated as:

$$\lambda_i(\boldsymbol{K}) = \lambda_i(\bar{\boldsymbol{K}}) + \sum_{m=1}^{R} \left.\frac{\partial \lambda_i}{\partial K_m}\right|_{\boldsymbol{K}=\bar{\boldsymbol{K}}} (K_m - \bar{K}_m) \; +$$

$$+ \frac{1}{2} \sum_{m=1}^{R} \sum_{n=1}^{R} \left.\frac{\partial^2 \lambda_i}{\partial K_m\, \partial K_n}\right|_{\boldsymbol{K}=\bar{\boldsymbol{K}}} (K_m - \bar{K}_m)(K_n - \bar{K}_n) \tag{1.7}$$

Introducing the notation:

$$\bar{\lambda}_i = \lambda_i(\bar{\boldsymbol{K}}) \tag{1.8}$$

$$\hat{\lambda}^I_{im} = \left.\frac{\partial \lambda_i}{\partial K_m}\right|_{\boldsymbol{K}=\bar{\boldsymbol{K}}} \tag{1.9}$$

$$\hat{\lambda}^{II}_{imn} = \left. \frac{\partial^2 \lambda_i}{\partial K_m \, \partial K_n} \right|_{\boldsymbol{K}=\bar{\boldsymbol{K}}} \tag{1.10}$$

and expressing the random perturbations from the mean values of the stiffness coefficients as:

$$\alpha_m = K_m - \bar{K}_m \tag{1.11}$$

equation (1.7) can be written as:

$$\lambda_i = \bar{\lambda}_i + \sum_{m=1}^{R} \hat{\lambda}^{I}_{im} \, \alpha_m + \frac{1}{2} \sum_{m=1}^{R} \sum_{n=1}^{R} \hat{\lambda}^{II}_{imn} \, \alpha_m \, \alpha_n \tag{1.12}$$

Proceeding in a similar way, the second order Taylor series expansion for the eigenvectors can be written as:

$$\boldsymbol{\phi}_i = \bar{\boldsymbol{\phi}}_i + \sum_{m=1}^{R} \hat{\boldsymbol{\phi}}^{I}_{im} \, \alpha_m + \sum_{m=1}^{R} \sum_{n=1}^{R} \hat{\boldsymbol{\phi}}^{II}_{imn} \, \alpha_m \, \alpha_n \tag{1.13}$$

where:

$$\bar{\boldsymbol{\phi}}_i = \boldsymbol{\phi}_i(\bar{\boldsymbol{K}}) \tag{1.14}$$

$$\hat{\boldsymbol{\phi}}^{I}_{im} = \boldsymbol{\phi}_i \left. \frac{\partial \boldsymbol{\phi}_i}{\partial K_m} \right|_{\boldsymbol{K}=\bar{\boldsymbol{K}}} \tag{1.15}$$

$$\hat{\boldsymbol{\phi}}^{II}_{imn} = \left. \frac{\partial^2 \boldsymbol{\phi}_i}{\partial K_m \, \partial K_n} \right|_{\boldsymbol{K}=\bar{\boldsymbol{K}}} \tag{1.16}$$

In the previous chapter we defined the Taylor series expansion for the eigenvalues and eigenvectors in terms of the so-called fixity factors μ_i . These non-dimensional factors allowed us to obtain compact expressions for the coefficients involved in the calculation of the rates of change of the eigenproperties. At first sight, it seems convenient here to make use of the same expansions. However, recalling that the fixity factors are defined in terms of the joint stiffness as:

$$\mu_i = \frac{1}{1 + 3\frac{EI}{LK_i}} \tag{1.17}$$

we observe that if the variables K_i are considered to be normal random variables, the same assumption cannot be extended to the factors μ_i , since equation (1.17) does not define a linear transformation. Therefore, if we are interested in maintaining the Gaussian distribution assumption to take advantage of its properties, the expansions have to be written in terms of the variables α_i . However, we can still employ the available expressions for the rates of change of the eigenproperties making use of the chain rule:

$$\hat{\lambda}^{I}_{im} = \frac{\partial \lambda_i}{\partial K_m} = \lambda^{I}_{im} \frac{\partial \mu_m}{\partial K_m} \tag{1.18}$$

$$\hat{\phi}^{I}_{i_m} = \frac{\partial \phi_i}{\partial K_m} = \phi^{I}_{i_m} \frac{\partial \mu_m}{\partial K_m} \tag{1.19}$$

$$\hat{\lambda}^{II}_{imn} = \frac{\partial^2 \lambda_i}{\partial K_m \, \partial K_n} \begin{cases} = \lambda^{II}_{imm} \left(\frac{\partial \mu_m}{\partial K_m}\right)^2 + \lambda^{I}_{im} \frac{\partial^2 \mu_m}{\partial K_m^2} & \text{for } m = n \\ = \lambda^{II}_{imm} \frac{\partial \mu_m}{\partial K_m} \frac{\partial \mu_m}{\partial K_n} & \text{for } m \neq n \end{cases} \tag{1.20}$$

$$\hat{\phi}^{II}_{imn} = \frac{\partial^2 \lambda_i}{\partial K_m \, \partial K_n} \begin{cases} = \phi^{II}_{imn} \left(\frac{\partial \mu_m}{\partial K_m}\right)^2 + \phi^{I}_{im} \frac{\partial^2 \mu_m}{\partial K_m^2} & \text{for } m = n \\ = \phi^{II}_{imn} \frac{\partial \mu_m}{\partial K_m} \frac{\partial \mu_m}{\partial K_n} & \text{for } m \neq n \end{cases} \tag{1.21}$$

in which:

$$\frac{\partial \mu_m}{\partial K_m} = \frac{3 \frac{EI}{L}}{\left(K_m + 3\frac{EI}{L}\right)^2} \tag{1.22}$$

$$\frac{\partial^2 \mu_m}{\partial K_m^2} = \frac{-6 \frac{EI}{L}}{\left(K_m + 3\frac{EI}{L}\right)^3} \tag{1.23}$$

To obtain the expected values of the $i-th$ eigenproperties, it is necessary to apply the expected value operator to the expansions (1.8- 1.11). Considering equation (1.2), we can write:

$$E\{\lambda_i\} = \bar{\lambda}_i + \frac{1}{2} \sum_{m=1}^{R} \sum_{n=1}^{R} \hat{\lambda}^{II}_{imn} \, E\{\alpha_m \alpha_n\} \tag{1.24}$$

$$E\{\phi i\} = \bar{\phi} i + \frac{1}{2} \sum_{m=1}^{R} \sum_{n=1}^{R} \hat{\phi} i^{II}_{mn} \, E\{\alpha_m \alpha_n\} \tag{1.25}$$

From these equations, we can see that the expected values differ from the zero-order terms that are solution of the averaged problem, and the difference is a linear function of the covariances of the random variables.

If the random variables are considered to be uncorrelated, then it is possible to write:

$$E\{\alpha_m \, \alpha_n\} = \delta_{mn} \, \sigma^2_m \tag{1.26}$$

where δ_{mn} is the Kroenecker delta, and σ^2_m is the variance of the random variable α_m . With this assumption, equations (1.22- 1.23) take the form:

$$E\{\lambda_i\} = \bar{\lambda}_i + \frac{1}{2} \sum_{m=1}^{R} \hat{\lambda}^{II}_{imm} \, \sigma^2_m \tag{1.27}$$

$$E\{\phi_i\} = \bar{\phi}_i + \frac{1}{2} \sum_{m=1}^{R} \hat{\phi}^{II}_{imm} \, \sigma^2_m \tag{1.28}$$

The variance of the $i-th$ eigenvalue is given by:

$$\sigma^2_{\lambda i} = E\{(\lambda i - E\{\lambda i\})^2\} \tag{1.29}$$

Substituting the corresponding expressions for λ_i and $E\{\lambda_i\}$ given by equations (1.12) and (1.4), respectively, and considering the linearity of the expected value operator, we can write:

$$\sigma^2_{\lambda_i} = E\left\{\left(\sum_{m=1}^{R} \hat{\lambda}^{I}_{im}\, \alpha_m\right)^2\right\} + \frac{1}{4} E\left\{\left(\sum_{m=1}^{R}\sum_{n=1}^{R} \hat{\lambda}^{II}_{imn}\, \alpha_m\, \alpha_n\right)^2\right\} +$$

$$+\frac{1}{4} E\left\{\left(\sum_{m=1}^{R}\sum_{n=1}^{R} \hat{\lambda}^{II}_{imn}\, E\{\alpha_m\, \alpha_n\}\right)^2\right\} +$$

$$+E\left\{\sum_{m=1}^{R}\sum_{r=1}^{R}\sum_{s=1}^{R} \hat{\lambda}^{I}_{im}\, \hat{\lambda}^{II}_{irs}\, \alpha_m\, \alpha_r\, \alpha_s\right\} -$$

$$-E\left\{\sum_{m=1}^{R}\sum_{r=1}^{R}\sum_{s=1}^{R} \hat{\lambda}^{I}_{im}\, \hat{\lambda}^{II}_{irs}\, \alpha_m\, E\{\alpha_r\, \alpha_s\}\right\} -$$

$$-\frac{1}{2} E\left\{\sum_{m=1}^{R}\sum_{n=1}^{R}\sum_{r=1}^{R}\sum_{s=1}^{R} \hat{\lambda}^{II}_{imn}\, \hat{\lambda}^{II}_{irs}\, \alpha_m\, \alpha_n\, E\{\alpha_r\, \alpha_s\}\right\} \tag{1.30}$$

If we consider the properties of normal zero-mean random variables, we are lead to the conclusion that the fourth and fifth terms in the above expression must vanish, since they involve odd moments:

$$E\{\alpha_m\, \alpha_r\, \alpha_s\} = 0 \tag{1.31}$$

$$E\{\alpha_m\, E\{\alpha_r\, \alpha_s\}\} = E\{\alpha_m\}\, E\{\alpha_r\, \alpha_s\} = 0 \tag{1.32}$$

and equation (1.28) reduces to:

$$\sigma^2_{\lambda_i} = \sum_{m=1}^{R}\sum_{n=1}^{R} \hat{\lambda}^{I}_{im}\, \hat{\lambda}^{I}_{in}\, E\{\alpha_m\, \alpha_n\} -$$

$$-\frac{1}{4}\sum_{m=1}^{R}\sum_{n=1}^{R}\sum_{r=1}^{R}\sum_{s=1}^{R} \hat{\lambda}^{II}_{imn}\, \hat{\lambda}^{II}_{irs}\, E\{\alpha_m\, \alpha_n\}\, E\{\alpha_r\, \alpha_s\} +$$

$$+\frac{1}{4}\sum_{m=1}^{R}\sum_{n=1}^{R}\sum_{r=1}^{R}\sum_{s=1}^{R} \hat{\lambda}^{II}_{imn}\, \lambda^{II}_{irs}\, E\{\alpha_m\, \alpha_n\, \alpha_r\, \alpha_s\} \tag{1.33}$$

The joint moment of four normal random variables can be written in terms of lower order moments as follows:

$$E\{\alpha_m\,\alpha_n\,\alpha_r\,\alpha_s\} = E\{\alpha_m\,\alpha_n\}\;E\{\alpha_r\,\alpha_s\} + E\{\alpha_m\,\alpha_r\}\;E\{\alpha_n\,\alpha_s\} +$$

$$+\;E\{\alpha_m\,\alpha_s\}\;E\{\alpha_n\,\alpha_r\} \tag{1.34}$$

and the multiple summation in the last term of equation (1.30) can be expressed in the following form:

$$\sum_{m=1}^{R}\sum_{n=1}^{R}\sum_{r=1}^{R}\sum_{s=1}^{R}\hat{\lambda}^{II}_{imn}\,\lambda^{II}_{irs}\,E\{\alpha_m\,\alpha_n\,\alpha_r\,\alpha_s\} =$$

$$= \sum_{m=1}^{R}\sum_{n=1}^{R}\sum_{r=1}^{R}\sum_{s=1}^{R}\Big(\hat{\lambda}^{II}_{imn}\,\hat{\lambda}^{II}_{irs} + \hat{\lambda}^{II}_{imr}\,\hat{\lambda}^{II}_{ins} +$$

$$+\,\hat{\lambda}^{II}_{ims}\,\hat{\lambda}^{II}_{inr}\Big)\;E\{\alpha_m\,\alpha_n\}\;E\{\alpha_r\,\alpha_s\} \tag{1.35}$$

Substituting equation (1.33) into equation (1.30), we finally obtain:

$$\sigma^2_{\lambda_i} = \sum_{m=1}^{R}\sum_{n=1}^{R}\hat{\lambda}^{I}_{im}\,\hat{\lambda}^{I}_{in}\,E\{\alpha_m\,\alpha_n\} +$$

$$+\,\frac{1}{4}\sum_{m=1}^{R}\sum_{n=1}^{R}\sum_{r=1}^{R}\sum_{s=1}^{R}\Big(\hat{\lambda}^{II}_{imr}\,\hat{\lambda}^{II}_{ins} + \hat{\lambda}^{II}_{ims}\,\hat{\lambda}^{II}_{inr}\Big)\;E\{\alpha_m\,\alpha_n\}\;E\{\alpha_r\,\alpha_s\} \tag{1.36}$$

The expression for the eigenvector variance can be obtained following an analogous derivation:

$$\sigma^2_{\phi_i} = \sum_{m=1}^{R}\sum_{n=1}^{R}\hat{\phi}^{I}_{im}\,\hat{\phi}^{I}_{in}\;E\{\alpha_m\,\alpha_n\} +$$

$$+\,\frac{1}{4}\sum_{m=1}^{R}\sum_{n=1}^{R}\sum_{r=1}^{R}\sum_{s=1}^{R}\Big(\hat{\phi}^{II}_{imr}\,\hat{\phi}^{II}_{ins} + \hat{\phi}^{II}_{ims}\,\hat{\phi}^{II}_{inr}\Big)\;E\{\alpha_m\,\alpha_n\}\;E\{\alpha_r\,\alpha_s\} \tag{1.37}$$

If we assume that the random variables are uncorrelated, then equations (1.34) and (1.35) reduce to:

$$\sigma^2_{\lambda_i} = \sum_{m=1}^{R}\left(\hat{\lambda}^{I}_{im}\right)^2\sigma^2_m + \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R}\left(\hat{\lambda}^{II}_{imn}\right)^2\sigma^2_m\,\sigma^2_n \tag{1.38}$$

$$\sigma^2_{\phi_i} = \sum_{m=1}^{R}\left(\hat{\phi}^{I}_{im}\right)^2\sigma^2_m + \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R}\left(\hat{\phi}^{II}_{imn}\right)^2\sigma^2_m\,\sigma^2_n \tag{1.39}$$

1.4 Numerical Examples

In the first part of this section we will validate the expressions for the first and second order moments of the eigenproperties derived in this chapter. For this we will consider the plane frame illustrated in Figure (2.4) and we will compare the values of the moments obtained with the *2nd* order perturbation expansion method against the results of a Monte-Carlo simulation. The Monte- Carlo simulation (hereafter referred to as MCS) is a very powerful method to determine the performance of systems with random parameters. The method consists in the generation of a set of systems derived from the original system by assigning values to the random parameters. All the systems are later statistically processed using techniques of sampling and parameter estimation [4,5]. The values assigned to the parameters of the systems are obtained by means of an algorithm that generates random numbers. This requires to select the probability distribution function that governs the behavior of these parameters.

Once the accuracy of the expressions obtained with the Second Order Perturbation Method (from now on designated as SOPM) is established, they will be used to obtain the probability density function of the eigenvalues and elements of the eigenvector. This will allow us to assess the level of dispersion in the dynamic properties of the structure introduced by the random variation of the stiffness coefficients. The two-level frame in Figure (2.16) will be used for this purpose.

1.4.1 Example No. 1: One-Story Frame

The subroutine GGNSM of the STAT-IMSL library [6] was used to generate two sets of random numbers. Both sets of numbers have a covariance matrix equal to the identity matrix. A pair of 50 uncorrelated standard normal random deviates ($\{R_i, Q_i\}, i = 1,2,...,50$) with zero mean value and variance equal to one were generated in this way. It is well known that normal random variables remain normal under linear transformations. Therefore, these variables can be used to generate another set of random normal variables ($\{k_{1_i}, k_{2_i}\}, i = 1,2,...,50$) with mean value equal to $\bar{k}$ and standard deviation σ_k . The relationship between the two sets is:

$$k_{1_i} = \sigma_k R_i + \bar{k} = (\rho_k R_i + 1)\bar{k} \quad ; \quad i = 1,2,...,50 \tag{1.40}$$

$$k_{2_i} = \sigma_k Q_i + \bar{k} = (\rho_k Q_i + 1)\bar{k} \quad ; \quad i = 1,2,...,50 \tag{1.41}$$

Fifty pairs of numbers representing the stiffness coefficients of the rotational springs at the beam's ends were generated in this way. The mean value of the coefficients k_{1_i} , k_{2_i} is such that the corresponding fixity factor is 0.50. Two different sets of pair of coefficients were generated for two values of the coefficient of variation (c.o.v.) of the stiffness parameters, ρ_k = 0.10 and ρ_k = 0.20.

Tables (1.1-6) present the mean values and standard deviations for the eigenvalues and eigenvectors of the first two modes of the frame in Figure (1.1) calculated with the two approaches. The frame was modelled with only three elements and 6 dof because the aim of the analysis is to compare the accuracy of the perturbation method without being concerned about the accuracy of the calculated eigenproperties. Tables (1.1) and (1.2) show the expected value and standard

deviation of the lower eigenvalue of the frame, respectively. The results shown under the Monte-Carlo column are the sample mean and sample standard deviation. All the tables include the limiting values corresponding to a confidence interval of 95%. It can be seen that the results obtained via the SOPM always lie within the confidence interval. Note also that according to the table, doubling the standard deviations of the connection stiffness produces the same effect on the standard deviations of the eigenvalues. The same phenomenum can be observed in the c.o.v. of the eigenvalues in Table (1.3). This reveals an almost linear probabilistic dependence between the stiffness of the connections and the eigenvalues of the structural system. Examining Table (1.3) it becomes apparent that the structural system filters out the effect that the uncertainties in the stiffness of the connections have on the eigenvalues of the system. For instance, when the c.o.v. of the stiffness of the joints is 0.10, the c.o.v. of the first and second eigenvalue is 0.0139 and 0.0378, respectively.

Tables (1.4-6) show the comparison in the statistics of the first eigenvector calculated using the SOPM and the MCS technique for two values of the c.o.v. of the connection stiffness. It is illustrative to discuss some characteristics of the results presented in these tables. The expression for the expected value of the $i-th$ eigenvector given by equation (1.28), includes the $2nd$ order derivatives of the eigenvector. It is shown in Chapter 3 of ref.[7] that these $2nd$ order derivatives are obtained in terms of a linear combination of the eigenvectors of the deterministic system associated with the mean values of the stiffness coefficients. Due to the symmetric configuration of the structure under consideration, these deterministic eigenvectors (or modal shapes) are either symmetric or antisymmetric. Thus, the expected values of the eigenvectors of the random system are also either symmetric or antisymmetric. The same situation occurs with the variance of the eigenvectors. The expression for the variance of the $i-th$ eigenvector is given by equation (1.39), and it includes the squares of the 1st order derivatives and the squares of the $2nd$ order derivatives. Therefore, for this particular example the variance vectors are also symmetric. On the other hand, the results of MCS are obtained from the statistical processing of a set of structures which are not necessarily symmetric, since the stiffness coefficients are uncorrelated random variables. Therefore, for a sample of finite size the expected values and variances of the eigenvectors do not necessarily show a strict symmetry. In spite of this fact, the mean values and standard deviations of all the elements of the eigenvector calculated using the SOPM fall inside the 95% confidence interval. The c.o.v. for the first mode of vibration are presented in Table (4.6). The table shows that the modal dof associated with the horizontal displacement of the ends of the beam (1st and 4th elements) are practically insensitive to the uncertainties in the stiffness of the connections. Moreover, there is an almost linear statistical relation between the components of the eigenvector and the stiffness of the joints. In this as well as in the previous tables the agreement between the perturbation-based results and the Monte-Carlo technique is remarkable.

1.4.2 Example No. 2: Two-Story Plane Frame

The next example is the two-story plane frame modelled with 42 dof shown in Figure 1.2. The stiffness coefficients of the connections at the ends of the beams at the first level, denoted as (a),(b),(c) and (d), are considered as uncorrelated normal random variables. They have a distri-

bution such that the mean value of the fixity factors is 0.70. Two different c.o.v., 0.10 and 0.20, will be used to describe the dispersion in the values of the stiffness coefficients.

The probability density function (PDF) corresponding to the stiffness coefficient of one of the flexible joint for three c.o.v. is shown in Figure (1.3). The multivariate density function corresponding to the four random stiffness coefficients is obtained as the product of the individual PDF since the random variables are assumed to be uncorrelated. Figure (1.4) shows the PDF for the case of two random stiffness coefficients.

If the stiffness coefficients have a normal distribution, the eigenvalues of the system will also be normally distributed. The PDF of the first eigenvalue is shown in Figure (1.5). The mean value and the standard deviation to plot the curve were calculated with the SOPM. The eigenvalues in the horizontal axis were divided by the corresponding eigenvalue obtained from the deterministic eigenproblem using the mean value of the stiffness coefficients. The c.o.v. for the random eigenvalue are 0.0078 and 0.0158 when the c.o.v. of the stiffness coefficients are 0.10 and 0.20, respectively. This indicates that, as in the previous example, the structure is not very sensitive to the uncertainties in the stiffness of the non-rigid joints. Figures (1.6) to (1.8) show the PDF for the *2nd*, *3rd* and *4th* eigenvalues. The *3rd* eigenvalue is the most sensitive to the random variability of the connections' stiffness. For c.o.v. of the stiffness equal to 0.10 and 0.20, the respective c.o.v. of the *3rd* eigenvalue are 0.0146 and 0.0296. It can be noted that the *2nd* and *4th* eigenvalues do not have a significant dispersion meaning that they are not much affected by the uncertainties in the stiffness of the connections at the first floor.

The PDFs for selected elements of the lower four eigenvectors are displayed in Figures (1.9-12). The results for the *1st* eigenvector are shown in Figure (1.9). The random connections have a more pronounced effect in the modal dof associated with the vertical displacement of the node in the beam $c-d$. The two PDF's for this dof depicted in the figure have c.o.v. equal to 0.3072 and 0.6172 for c.o.v. of the stiffness equal 0.10 and 0.20, respectively. Although the effect is not as pronounced as in the previous case, the uncertainties in the connections also have an important influence in the rotation of joint (d). The PDF's associated with this dof plotted in the figure have c.o.v. equal to 0.0253 and 0.0517. On the contrary, the horizontal displacement of node (d) is relatively insensitive to the random connections, as evidenced by the sharp PDF for this dof shown in the figure.

Figure (1.10) show similar results but this time for the *2nd* eigenvector. The PDF's are similar to those for the *1st* eigenvector. The PDF's for the modal displacements corresponding to the *3rd* eigenvalue are presented in Figure (1.11). In this case, the largest dispersion is associated with the horizontal displacement of node (d). The c.o.v. are 0.0239 and 0.0485 when the respective coefficients for the stiffness are 0.10 and 0.20. Finally, Figure (1.12) displays the results for the *4th* eigenvector. Here the vertical displacement of the interior node on beam (c)-(d) is the modal dof most affected by the randomness of the connections. The distributions have c.o.v. equal to 0.0712 and 0.1425 when the distribution of the stiffness have c.o.v. equal to 0.10 and 0.20, respectively.

Table 1.1: Eigenvalue Statistics: Mean Values.

Eigenvalue #	2nd. Order Perturbation	Montecarlo Mean	95% Confidence: Lower Limit	95% Confidence: Upper Limit
a) Coefficient of Variation = 0.10				
1	25906.5706	25946.4363	25843.2059	26047.6647
2	441811.7583	439868.6662	435146.3028	444591.0357
b) Coefficient of Variation = 0.20				
1	25815.9075	25887.3222	25678.1778	26096.4666
2	444647.1245	440836.0432	431219.2278	450450.8586

Table 1.2: Eigenvalue Statistics: Standard Deviations.

Eigenvalue #	2nd. Order Perturbation	Montecarlo St. Deviation	95% Confidence: Lower Limit	95% Confidence: Upper Limit
a) Coefficient of Variation = 0.10				
1	343.1863	359.7086	300.4779	448.2445
2	15936.6957	16616.3286	13860.2254	20706.1342
b) Coefficient of Variation = 0.20				
1	664.9025	736.9047	614.7260	917.0342
2	32303.0190	33634.6361	28283.3047	42162.4117

Table 1.3: Eigenvalue Statistics: Coefficients of Variation.

Eigenvalue #	2nd. Order Perturbation	Montecarlo
a) Coefficient of Variation = 0.10		
1	0.0132	0.0139
2	0.0361	0.0378
b) Coefficient of Variation = 0.20		
1	0.0269	0.0284
2	0.0728	0.0768

Table 1.4: First Eigenvector Statistics: Mean Values.

DOF #	2nd. Order Perturbation	Montecarlo Mean	95% Confidence: Lower Limit	95% Confidence: Upper Limit
a) Coefficient of Variation = 0.10				
1	3.6399E-01	3.6396E-01	3.6391E-01	3.6401E-01
2	1.3977E-03	1.3990E-03	1.3769E-03	1.4212E-03
3	2.9631E-03	2.9607E-03	2.9511E-03	2.9704E-03
4	3.6399E-01	3.6396E-01	3.6390E-01	3.6400E-01
5	1.3977E-03	1.3986E-03	1.3862E-03	1.4108E-03
6	2.9631E-03	2.9584E-03	2.9482E-03	2.9686E-03
b) Coefficient of Variation = 0.20				
1	3.6400E-01	3.6392E-01	3.6383E-01	3.6402E-01
2	1.4111E-03	1.3940E-03	1.3487E-03	1.4393E-03
3	2.9704E-03	2.9562E-03	2.9467E-03	2.9658E-03
4	3.6400E-01	3.6391E-01	3.6381E-01	3.6401E-01
5	1.4111E-03	1.3769E-03	1.3352E-03	1.4185E-03
6	2.9704E-03	2.9620E-03	2.9410E-03	2.9829E-03

Table 1.5: First Eigenvector Statistics: Standard Deviations.

DOF #	2nd. Order Perturbation	Montecarlo St. Deviation	95% Confidence: Lower Limit	95% Confidence: Upper Limit
a) Coefficient of Variation = 0.10				
1	1.6773E-04	1.7189E-04	1.4358E-04	2.1419E-04
2	7.0381E-05	7.7280E-05	6.4555E-05	9.6302E-05
3	3.2871E-05	3.3559E-05	2.8033E-05	4.1819E-05
4	1.6773E-04	1.7911E-04	1.4962E-04	2.2320E-04
5	7.0381E-05	7.1828E-05	5.9999E-05	8.9505E-05
6	3.2871E-05	3.5570E-05	2.9713E-05	4.4325E-05
b) Coefficient of Variation = 0.20				
1	3.3603E-04	3.3784E-04	2.8221E-04	4.2099E-04
2	1.4205E-04	1.5776E-04	1.3178E-04	1.9659E-04
3	6.6444E-05	6.8073E-05	5.6864E-05	8.4826E-05
4	3.3603E-04	3.5319E-04	2.9504E-04	4.4013E-04
5	1.4205E-04	1.4521E-04	1.2130E-04	1.8095E-04
6	6.6444E-05	7.3115E-05	6.1076E-05	9.1112E-05

Table 1.6: First Eigenvector Statistics: Coefficients of Variation.

DOF #	2nd. Order Perturbation	Montecarlo
a) Coefficient of Variation = 0.10		
1	0.0005	0.0005
2	0.0504	0.0552
3	0.0111	0.0113
4	0.0005	0.0005
5	0.0504	0.0517
6	0.0111	0.0120
b) Coefficient of Variation = 0.20		
1	0.0009	0.0009
2	0.1007	0.1132
3	0.0224	0.0229
4	0.0009	0.0010
5	0.1007	0.1055
6	0.0224	0.0247

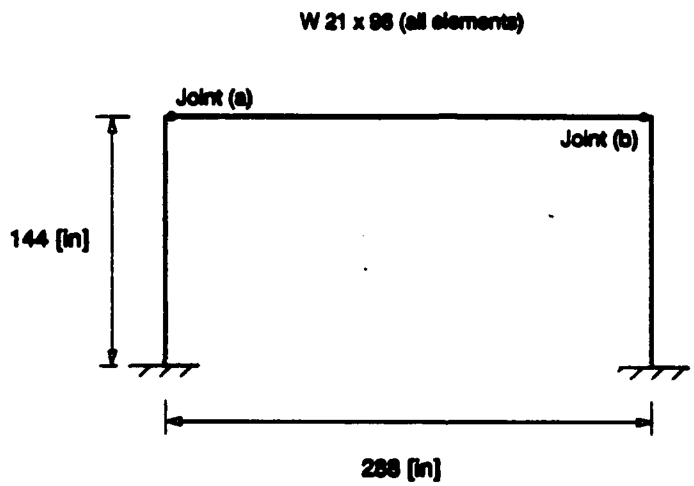

Figure 1.1: One Story Frame For Example No. 1.

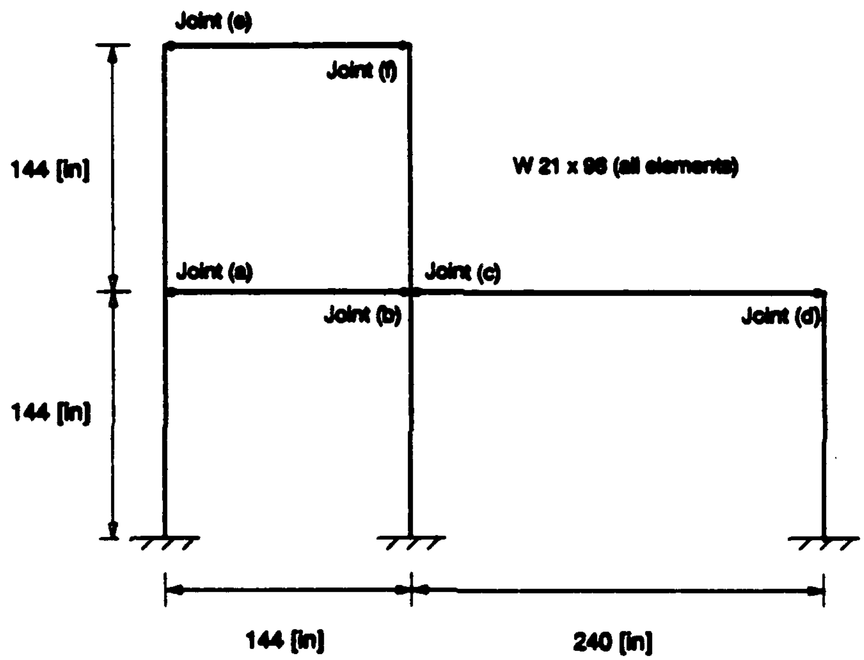

Figure 1.2: Two Story Frame For Example No. 2.

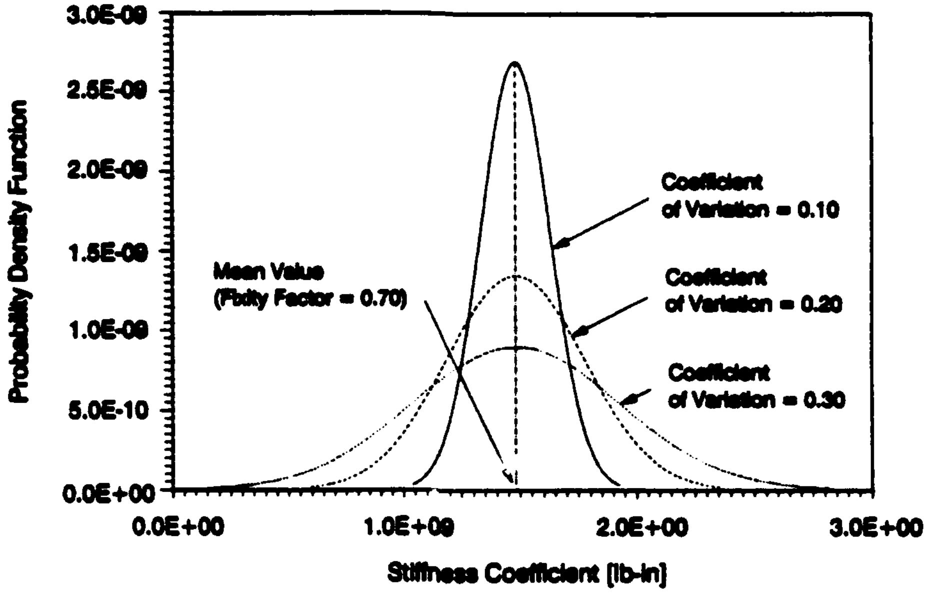

Figure 1.3: Probability Density Function of Stiffness Coefficient.

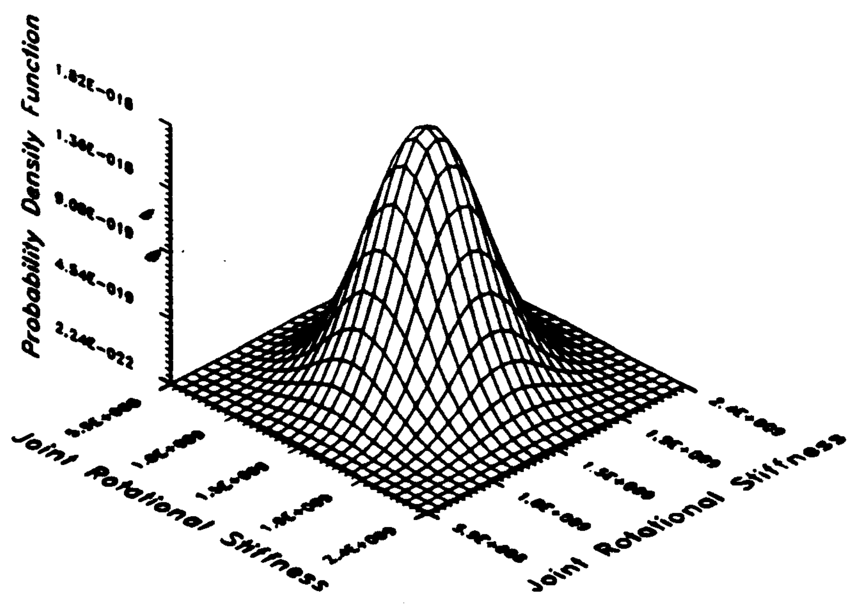

Figure 1.4: Bivariate Density Function of Stiffness Coefficients.

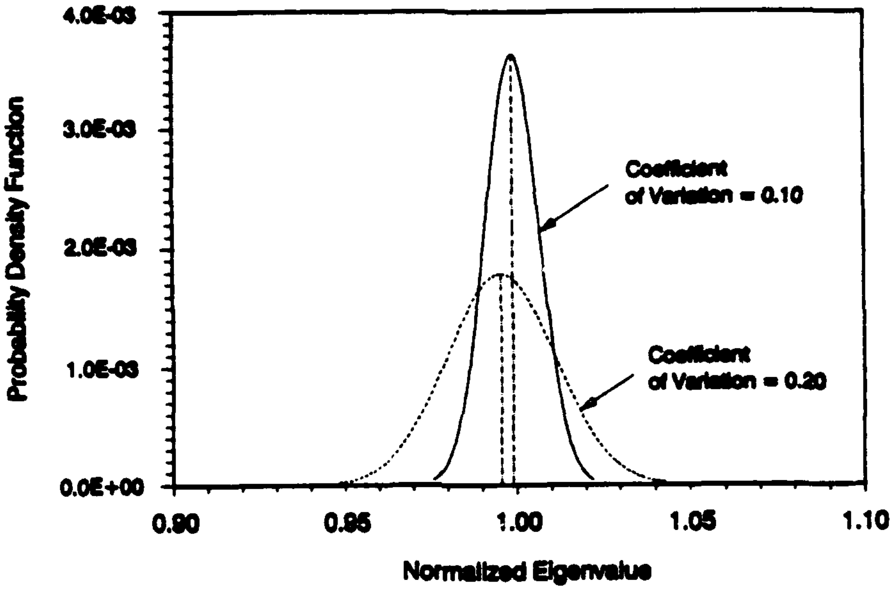

Figure 1.5: Probability Density Function of First Eigenvalue.

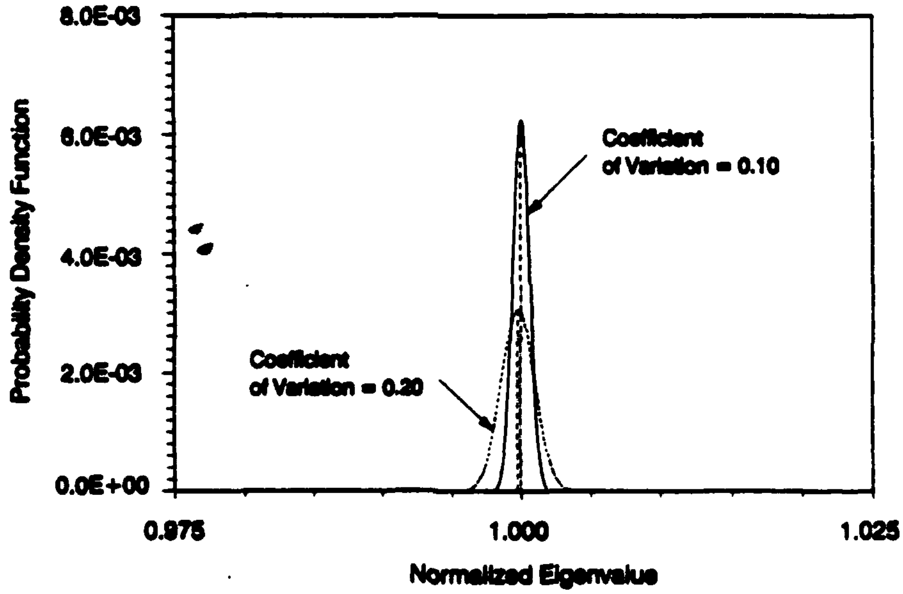

Figure 1.6: Probability Density Function of Second Eigenvalue.

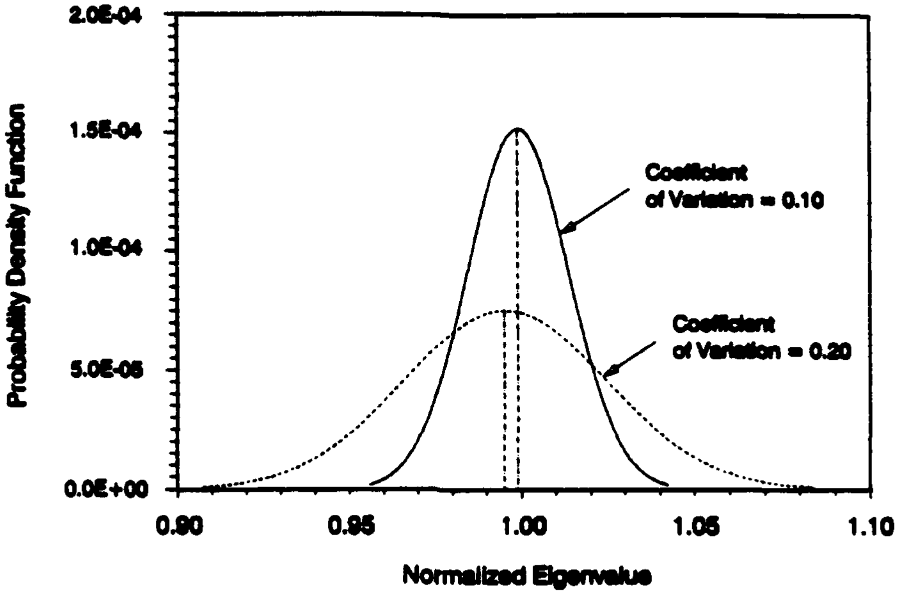

Figure 1.7: Probability Density Function of Third Eigenvalue.

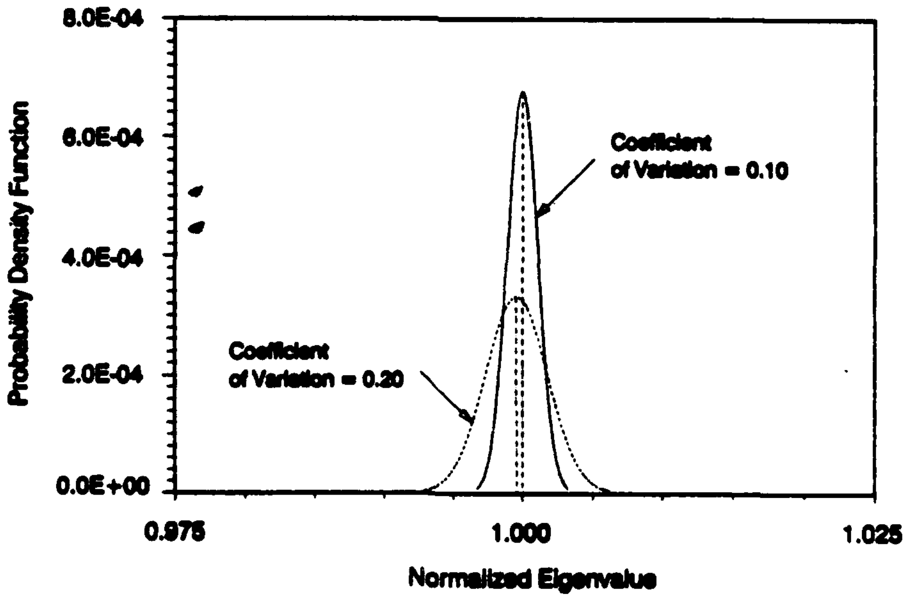

Figure 1.8: Probability Density Function of Fourth Eigenvalue.

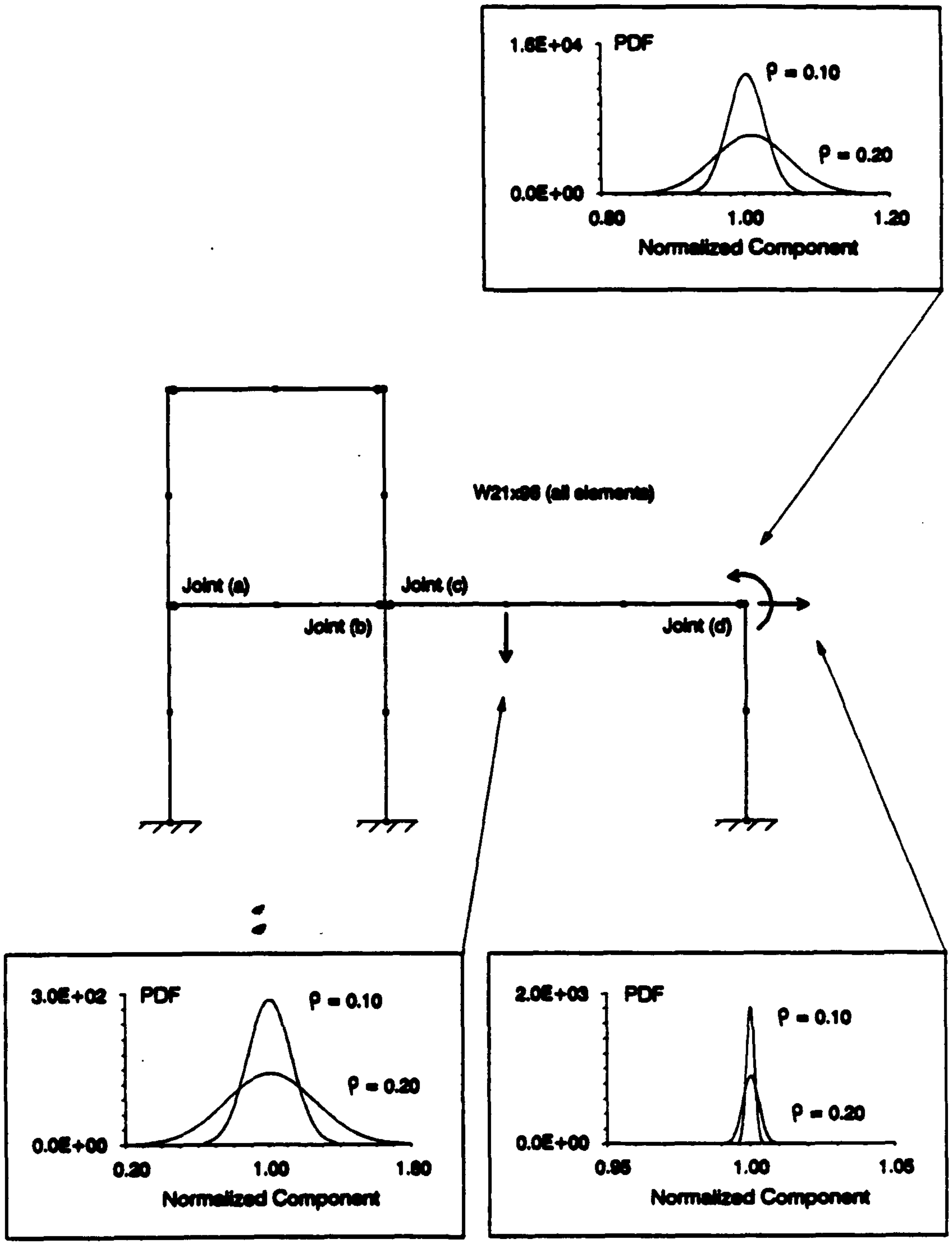

Figure 1.9: Probability Density Function of Selected Dof's of First Eigenvector.

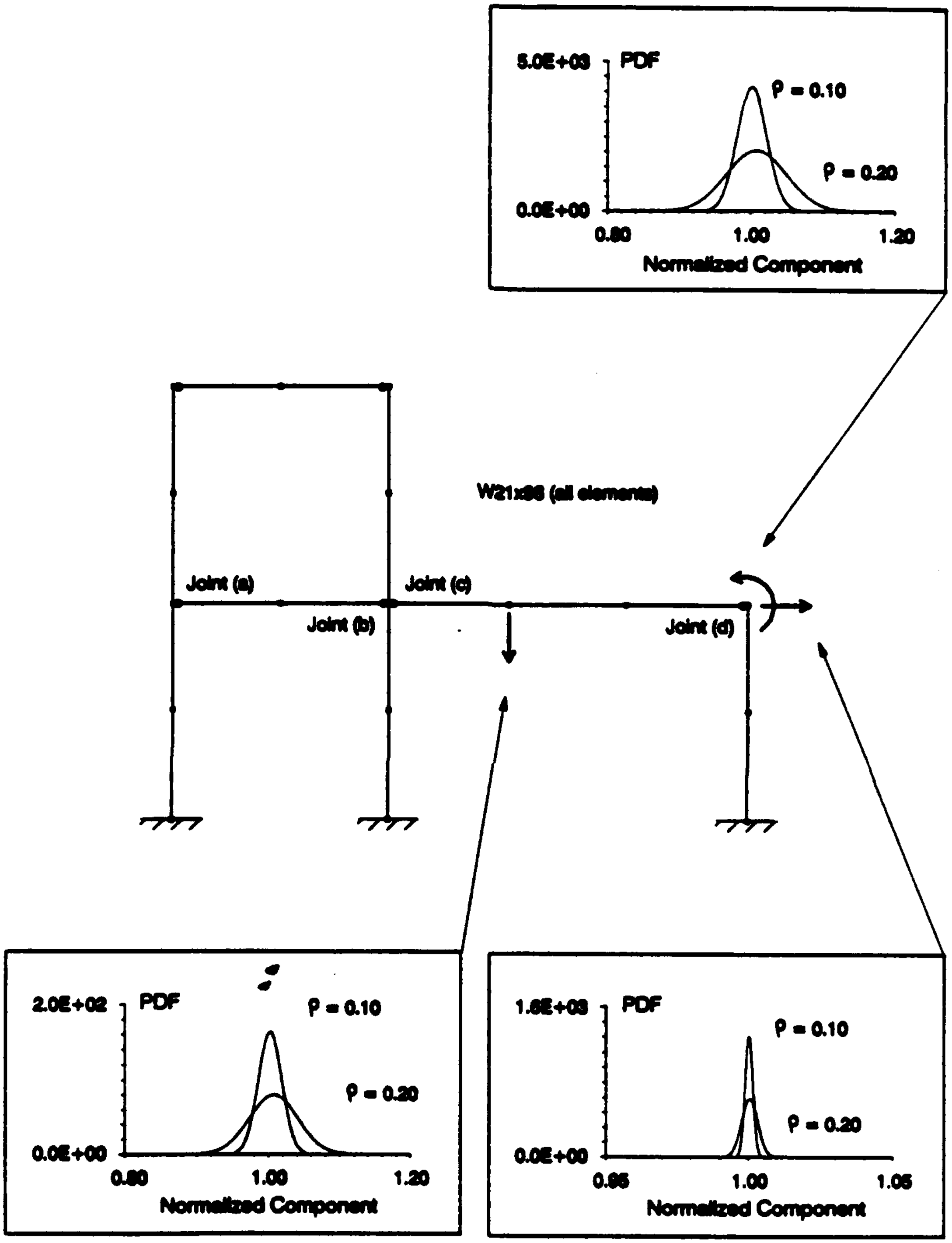

Figure 1.10: Probability Density Function of Selected Dof's of Second Eigenvector.

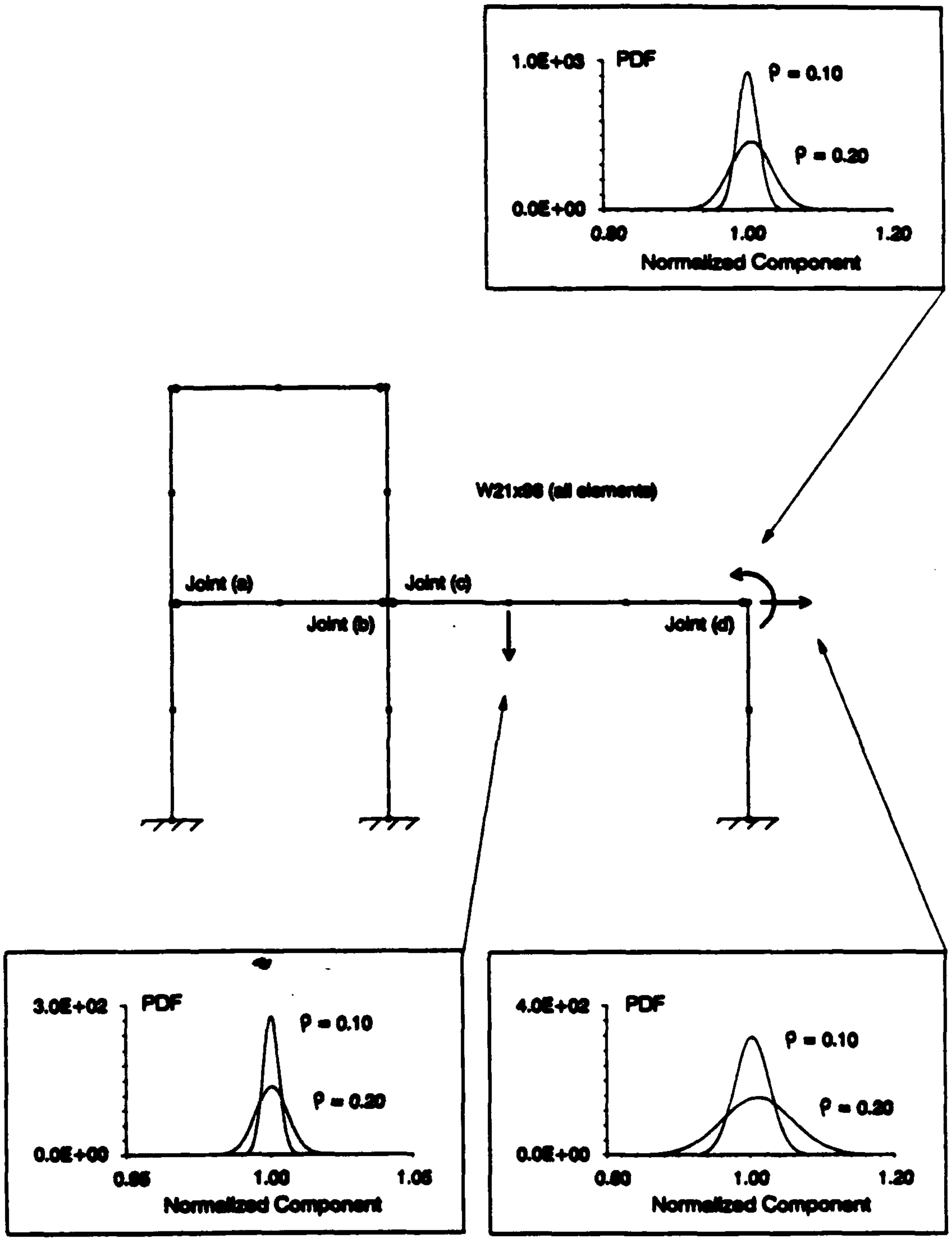

Figure 1.11: Probability Density Function of Selected Dof's of Third Eigenvector.

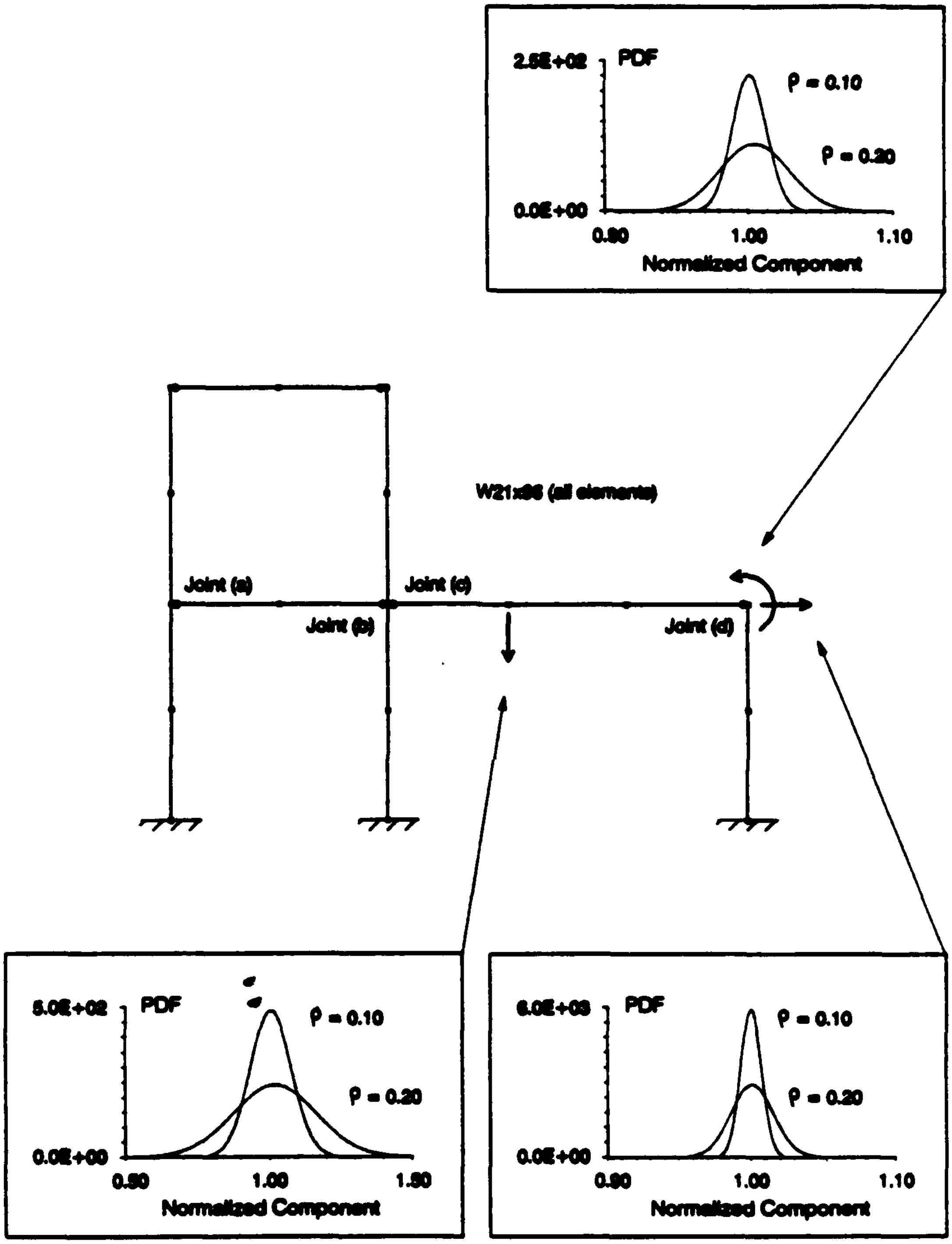

Figure 1.12: Probability Density Function of Selected Dof's of Fourth Eigenvector.

Chapter 2

Response of Structures with Random Parameters

2.1 Introduction

In the previous chapter we presented the application of the stochastic finite element method to determine the statistics of the natural frequencies and modes of structures with flexible random connections. In this chapter we will apply the same technique to calculate the statistics of the response of this type of structures when they are subjected to deterministic dynamic excitations. The level of complexity of the analysis is significantly increased due to the fact that the perturbation expansion that was used in the previous chapter may lead to a solution that is "*non-uniform*" or unbounded in time. The perturbation expansion previously used is usually referred to as the "*straightforward expansion*". The approximate solution generated by this method is only applicable in the initial instants of time due to unbounded terms appearing in the higher order terms of the expansion. To overcome the difficulties associated with this technique, a solution methodology based on the "*method of multiple scales*" will be developed.

First we will examine the deterministic and stochastic response of single dof systems in which the natural frequency and the excitation are functions of a set of parameters. The parameters will be considered as deterministic perturbations on a set of initial values in the first case, and as random variables in the second case. This study will not only allow us to comprehend the cause and significance of the unbounded terms but the results are important for the final goal, i.e, the response statistics of multi-dof systems with random time-invariant parameters. The latter study is undertaken in the last sections of this chapter. For multi-dof systems, the random parameters will be identified as the stiffness coefficients of the rotational springs representing the flexible connections. The case of a single and multi-dof system subjected to a simple dynamic loading will be examined in detail to illustrate the application of the straightforward expansion and the multiple scales approach.

2.2 Single Degree-of-Freedom Systems

Consider a single dof oscillator with natural frequency ω, and damping ratio ξ subjected to an excitation $f(t)$:

$$\ddot{\eta}(t) + 2\,\xi\,\omega\,\dot{\eta}(t) \;+\; \omega^2\,\eta(t) \;=\; f(t) \tag{2.1}$$

We will assume that the natural frequency of the oscillator and the excitation are function of R variables $\alpha_1, \alpha_2, ..., \alpha_R$. The reason for assuming this dependence will become apparent later when we calculate the statistics of the response. To shorten the notation we will group the parameters α_m in a single vector $\boldsymbol{\alpha}$. These parameters represent perturbations in the original values of ω and $f(t)$. Without loss of generality, the initial or unperturbed state of ω and $f(t)$ will be associated with the zero value of the parameters α_m . The response of the oscillator will also be a function of the parameters α_m and equation (2.1) can be written as:

$$\ddot{\eta}(\boldsymbol{\alpha},t) + 2\,\xi\,\omega\,\dot{\eta}(\boldsymbol{\alpha},t) \;+\; \omega^2\,\eta(\boldsymbol{\alpha},t) \;=\; f(\boldsymbol{\alpha},t) \tag{2.2}$$

2.2.1 The Straightforward Expansion

The objective of the analysis presented in this and in the following sections is to obtain the response $\eta(\boldsymbol{\alpha},t)$ without having to solve equation (2.2) and assuming that we know the unperturbed values of ω and $f(t)$. A straightforward perturbation expansion can be used for this purpose. The method of straightforward expansion is similar to the second order perturbation method used in Chapter 4 to calculate the statistics of the eigenvalues and eigenvectors. It consists in obtaining an approximate expression for the response by means of a Taylor series expansion in terms of the parameters α_m. Retaining only up to second order terms in the series we can write:

$$\eta(\boldsymbol{\alpha},t) = \eta_0(t) \;+\; \sum_{m=1}^{R} \eta_m^{I}(t)\,\alpha_m \;+\; \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R} \eta_{mn}^{II}(t)\,\alpha_m\,\alpha_n \tag{2.3}$$

where $\eta_0(t)$ is the response of the system when the parameters assume the initial or unperturbed values. The superscripts I and II represent, respectively, the first and second order rates of change of the function $\eta(\boldsymbol{\alpha},t)$ with respect to the parameters α_m evaluated at the initial state of the system. That is:

$$\eta_m^{I}(t) \;=\; \left.\frac{\partial\eta(\boldsymbol{\alpha},t)}{\partial\alpha_m}\right|_{\boldsymbol{\alpha}=0} \tag{2.4}$$

$$\eta_{mn}^{II}(t) \;=\; \left.\frac{\partial^2\eta(\boldsymbol{\alpha},t)}{\partial\alpha_m\,\partial\alpha_n}\right|_{\boldsymbol{\alpha}=0} \tag{2.5}$$

The derivatives of the response $\eta(\boldsymbol{\alpha},t)$ with respect to the parameters α_m are not known at this stage. The goal of the solution process is precisely to obtain these derivatives.

To simplify the notation in the subsequent developments the following notation will be introduced:

$$\eta_1(t) \;=\; \sum_{m=1}^{R} \eta_m^{I}(t)\,\alpha_m \tag{2.6}$$

$$\eta_2(t) \;=\; \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R} \eta_{mn}^{II}(t)\,\alpha_m\,\alpha_n \tag{2.7}$$

When using perturbation methods it is customary to introduce a coefficient ε to identify the order of the different terms involved in the expansions. Employing equations (2.6) and (2.7) and the perturbation parameter ε, the expansion (2.3) can then be written as follows:

$$\eta(\boldsymbol{\alpha},t) = \eta_0(t) + \varepsilon\,\eta_1(t) + \varepsilon^2\,\eta_2(t) \tag{2.8}$$

As it will become evident later, for the cases we are interested in, the natural frequency and the excitation in equation (2.2) are nonlinear functions of the parameters α_m . Except for trivial cases, it is not possible to obtain the exact closed form functional relationship. However, the frequency and the excitation can also be expanded in Taylor series around the initial values of the parameters α_m:

$$\omega(\boldsymbol{\alpha}) = \bar{\omega} + \sum_{m=1}^{R} \omega_m^I\,\alpha_m + \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R} \omega_{mn}^{II}\,\alpha_m\,\alpha_n \tag{2.9}$$

$$f(\boldsymbol{\alpha},t) = f_0(t) + \sum_{m=1}^{R} f_m^I(t)\,\alpha_m + \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R} f_{mn}^{II}(t)\,\alpha_m\,\alpha_n \tag{2.10}$$

where the first and second order rates of change of ω and $f(t)$ are defined as:

$$\omega_m^I = \left.\frac{\partial\omega}{\partial\alpha_m}\right|_{\boldsymbol{\alpha}=0} \quad ; \quad f_m^I(t) = \left.\frac{\partial f(t)}{\partial\alpha_m}\right|_{\boldsymbol{\alpha}=0} \tag{2.11}$$

$$\omega_{mn}^{II} = \left.\frac{\partial^2\omega}{\partial\alpha_m\,\partial\alpha_m}\right|_{\boldsymbol{\alpha}=0} \quad ; \quad f_{mn}^{II} = \left.\frac{\partial^2 f(t)}{\partial\alpha_m\,\partial\alpha_n}\right|_{\boldsymbol{\alpha}=0} \tag{2.12}$$

Introducing the following notation the first and second order correction terms:

$$\omega_1 = \sum_{m=1}^{R} \omega_m^I\,\alpha_m \quad ; \quad f_1(t) = \sum_{m=1}^{R} f_m^I(t)\,\alpha_m \tag{2.13}$$

$$\omega_2 = \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R} \omega_{mn}^{II}\,\alpha_m\,\alpha_n \quad ; \quad f_2(t) = \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R} f_{mn}^{II}\,\alpha_m\,\alpha_n \tag{2.14}$$

and using equations (2.13- 2.14) along with the perturbation parameter ε the expansions in equations (2.9- 2.10) can be written in the simpler form:

$$\omega(\boldsymbol{\alpha}) = \bar{\omega} + \varepsilon\,\omega_1 + \varepsilon^2\,\omega_2 \tag{2.15}$$

$$f(\boldsymbol{\alpha},t) = f_0(t) + \varepsilon\,f_1(t) + \varepsilon^2\,f_2(t) \tag{2.16}$$

Substituting equations (2.8) and (2.9-2.10) in (2.2) yields:

$$\left(\ddot{\eta}_0(t) + \varepsilon\,\ddot{\eta}_1(t) + \varepsilon^2\,\ddot{\eta}_2(t)\right) +$$

$$+ 2\xi \left(\bar{\omega} + \varepsilon\,\omega_1 + \varepsilon^2\omega_2\right) \left(\dot{\eta}_0(t) + \varepsilon\,\dot{\eta}_1(t) + \varepsilon^2\dot{\eta}_2(t)\right) +$$

$$+ \left(\bar{\omega} + \varepsilon\,\omega_1 + \varepsilon^2\omega_2\right)^2 \left(\eta_0(t) + \varepsilon\,\eta_1(t) + \varepsilon^2\eta_2(t)\right) =$$

$$= f_0(t) + \varepsilon\,f_1(t) + \varepsilon^2 f_2(t) \tag{2.17}$$

Collecting terms of the same order in the perturbation parameter, we obtain the following hierarchical equations:

$$O(\varepsilon^0): \qquad \ddot{\eta}_0(t) + 2\,\xi\,\bar{\omega}\,\dot{\eta}_0(t) + \bar{\omega}^2\,\eta_0(t) = f_0(t) \tag{2.18}$$

$$O(\varepsilon^1): \qquad \ddot{\eta}_1(t) + 2\,\xi\,\bar{\omega}\,\dot{\eta}_1(t) + \bar{\omega}^2\,\eta_1(t) = \tilde{f}_1(t) \tag{2.19}$$

$$O(\varepsilon^2): \qquad \ddot{\eta}_2(t) + 2\,\xi\,\bar{\omega}\,\dot{\eta}_2(t) + \bar{\omega}^2\,\eta_2(t) = \tilde{f}_2(t) \tag{2.20}$$

where:

$$\tilde{f}_1(t) = f_1(t) - 2\;xi\,\omega_1\,\dot{\eta}_0(t) - 2\,\bar{\omega}\,\omega_1\,\eta_0(t) \tag{2.21}$$

$$\tilde{f}_2(t) = f_2(t) - 2\,\xi\,\omega_2\,\dot{\eta}_0(t) - \left(2\,\bar{\omega}\,\omega_2 + \omega_1^2\right)\eta_0(t) - 2\,\xi\,\omega_1\,\dot{\eta}_1(t) - 2\,\bar{\omega}\,\omega_1\,\eta_1(t) \tag{2.22}$$

It is important to note that the excitation terms $\tilde{f}_i(t)$ for the first and second order equations contain, respectively, the solutions of the zero and first order equations. Since the differential operator in the three equations (2.18- 2.20) is the same, these excitation terms associated with the solutions of the lower order equations are a potential source of complications. Indeed, it is possible that resonance phenomena may occur in the higher order equations since the forcing functions for these equations are the solutions of the equations of lower order. In this case the solutions will have some terms that grow with time which are characteristic of resonant responses. In the terminology of perturbation methods these terms are called *secular terms*. As we discusse later, this problem will take place for almost all excitations.

To examine the phenomenon we proceed to study the solution of the hierarchical equations (2.18- 2.20). Some of the material in the sequel is well known from the elementary theory of ordinary differential equations. However, it is believed that its inclusion here is warranted because it is crucial for our discussion on the appearance of secular terms.

2.2.1.1 Solution of the Zeroth Order Equation

We begin by examining the zero order equation (2.18):

$$\ddot{\eta}_0(t) + 2\,\xi\,\bar{\omega}\,\dot{\eta}_0(t) + \bar{\omega}^2\,\eta_0(t) = f_0(t)$$

There are two ways to express the total solution of equation (2.18). One way is to write down the solution as the sum of an homogeneous or complementary solution $\eta_0^H(t)$ and an inhomogeneous or particular solution $\eta_0^{IH}(t)$:

$$\eta_0(t) = \eta_0^H(t) + \eta_0^{IH}(t) \tag{2.23}$$

The homogeneous solution has the form:

$$\eta_0^H(t) = a\,e^{-\xi\bar{\omega}t}\,Sin(\bar{\omega}_d t + \theta) \tag{2.24}$$

where a and θ are coefficients to be determined later and $\bar{\omega}_d$ is the damped natural frequency:

$$\bar{\omega}_d = \bar{\omega}\sqrt{1-\xi^2} \tag{2.25}$$

The inhomogeneous solution depends on the particular form of the right hand side term in equation (2.18). The solution $\eta_0^{IH}(t)$ can be obtained using, for example, the method of undetermined coefficients. The complete solution can then be written as:

$$\eta_0(t) = a\,e^{-\xi\bar{\omega}t}\,Sin(\bar{\omega}_d t + \theta) + \eta_0^{IH}(t) \tag{2.26}$$

The coefficients a and θ must be obtained by applying the initial conditions to the total response given by equation (2.26).

Alternatively, the solution of equation (2.18) can be written in terms of the response to initial conditions $\eta_0^{IC}(t)$ and the forced response $\eta_0^F(t)$:

$$\eta_0(t) = \eta_0^{IC}(t) + \eta_0^F(t) \tag{2.27}$$

The response to initial conditions $\eta_0^{IC}(t)$ is identical to equation (2.24):

$$\eta_0^H(t) = a_0\,e^{-\xi\bar{\omega}t}\,Sin(\bar{\omega}_d t + \theta_0) \tag{2.28}$$

with the difference that the coefficients a_0 and θ_0 have to be obtained by applying the initial conditions only to the solution defined by equation (2.28). The forced response $\eta_0^F(t)$ is expressed in terms of a convolution integral:

$$\eta_0^F(t) = \int_0^t f_0(\tau)\,h(t-\tau)\,d\tau \tag{2.29}$$

where $h(t)$ is the unit impulse response function which for an underdamped system is defined as:

$$h(t) = \frac{1}{\omega_d}e^{-\xi\bar{\omega}t}sin(\omega_d t) \tag{2.30}$$

Therefore, the total solution becomes:

$$\eta_0(t) = a_0\, e^{-\xi\bar{\omega}t}\, Sin(\bar{\omega}_d t + \theta_0) + \frac{1}{\omega_d}\int_0^t f_0(\tau)\, e^{-\xi\bar{\omega}(t-\tau)}\, Sin(\omega_d(t - \tau))\, d\tau \tag{2.31}$$

Based on the analysis of the results obtained so far, the following observations about the two solution methods and the potential difficulties associated with them can be made:

When the solution is expressed in the form of equations (2.26) or (2.31), the terms $\eta_0^H(t)$ or $\eta_0^{IC}(t)$ are harmonic functions with frequency equal to the natural frequency of the oscillator. Clearly, these terms will induce secular terms in the solution of the first order equations.

When the excitation is defined by a relatively simple expression, the response can be obtained using the method of equation (2.26). In most situations, there will not be any harmonic components with the oscillator's frequency included in the inhomogeneous solution $\eta_0^{IH}(t)$. Therefore, in this case the only source of secular terms will be the homogeneous solution $\eta_0^H(t)$. However, one must take into account that both solutions become interconnected when the total solution is used to satisfy the initial conditions.

When the integral in equation (2.31) is solved for a specific loading function, the resulting expression will have harmonic components with frequency equal to the oscillator's frequency. If the initial conditions are zero, this is the only term that will introduce secular terms.

2.2.1.2 Solution of the First Order Equations

To solve the first order equations we substitute equations (2.6) and (2.13) in equation (2.19), and equate those terms in both sides that are multiplied by the same variable α_m. The $m - th$ equation of first order can then be written as follows:

$$\ddot{\eta}_m^I(t) + 2\,\xi\,\bar{\omega}\,\dot{\eta}_m^I(t) + \bar{\omega}^2\,\eta_m^I(t) = \tilde{f}_m^I(t) \tag{2.32}$$

where:

$$\tilde{f}_m^I(t) = f_m^I(t) - 2\,\xi\,\omega_m^I\,\dot{\eta}_0(t) - 2\,\bar{\omega}\,\omega_m^I\,\eta_0(t) \tag{2.33}$$

Substituting in the above expression the zeroth order solution defined in equation (2.26) we obtain:

$$\tilde{f}_m^I(t) = -2\,\bar{\omega}\,\omega_m^I\,a\,e^{-\xi\bar{\omega}t}\left(\left(1-\xi^2\right)Sin(\bar{\omega}_d t + \theta) + \frac{\xi}{\sqrt{1-\xi^2}}Cos(\bar{\omega}_d\,t + \theta)\right) +$$

$$+ f_m^I(t) - 2\,\omega_m^I\left(\xi\,\dot{\eta}_0^{IH}(t) + \bar{\omega}\,\eta_0^{IH}(t)\right) \tag{2.34}$$

The solution of equation (2.32) can be written as the sum of the response due to the initial conditions and the forced response. For zero initial conditions, the latter term will manifest

the limitations of the method. Therefore, we will focus our attention on this term. The forced response can be obtained substituting in the Duhamel integral the excitation defined in equation (2.34):

$$\eta_m^I(t)^F = -2\,\bar{\omega}\,\omega_m^I\,a\left(1-\xi^2\right)\int_0^t e^{-\xi\bar{\omega}\tau}\,Sin\,(\bar{\omega}_d\tau + \theta)\;h(t-\tau)\,d\tau -$$

$$-2\,\bar{\omega}\,\omega_m^I\,a\,\frac{\xi}{\sqrt{1-\xi^2}}\int_0^t e^{-\xi\bar{\omega}\tau}\,Cos\,(\bar{\omega}_d\tau + \theta)\;h(t-\tau)\,d\tau +$$

$$+\int_0^t f_m^I(\tau) - 2\,\omega_m^I\,\left(\xi\,\dot{\eta}_0^{IH}(\tau) + \omega\,\eta_0^{IH}(\tau)\right)\,h(t-\tau)\,d\tau \qquad (2.35)$$

Solving the first two integrals the forced response becomes:

$$\eta_m^I(t)^F = 4\,\bar{\omega}\,\omega_m^I\,a\,e^{-\xi\bar{\omega}t}\,[p\,\bar{\omega}_d t\,Cos\,(\bar{\omega}_d t + \theta) - r\,\bar{\omega}_d t\,Sin\,(\bar{\omega}_d t + \theta) -$$

$$-\,Sin(\bar{\omega}_d t)\,\left(p\,Cos(\theta) - r\,Sin(\theta)\right)] +$$

$$+\int_0^t f_m^I(\tau) - 2\,\omega_m^I\,\left(\xi\,\dot{\eta}_0^{IH}(\tau) + \omega\,\eta_0^{IH}(\tau)\right)h(t-\tau)\,d\tau \qquad (2.36)$$

where the coefficients p and r are:

$$p = 1-\xi^2 \qquad (2.37)$$

$$r = \frac{\xi}{\sqrt{1-\xi^2}} \qquad (2.38)$$

Note that if damping is neglected, the first two terms inside the brackets in equation (2.36) become unbounded due to the terms linear in the variable t.

2.2.1.3 Deterministic Straightforward Expansion for the Undamped Oscillator Subjected to a Step Force

In order to gain insight into the peculiarities of the approximate solutions obtained via perturbation techniques, the specific case of an undamped oscillator subjected to a step load of magnitude Fo will be considered. The oscillator is assumed to be stationary when the load is applied. The equation of motion assumes the simple form:

$$\ddot{\eta}(\alpha,t) + \omega(\alpha)^2\,\eta(\alpha,t) = F_0 \qquad (2.39)$$

To express the solution by means of the perturbation expansion in equation (2.8) we need to solve the hierarchy of equations (2.18- 2.20). To express the solution of the zeroth order

equation (2.18) in the form of equation (2.26) we need to define the particular solution $\eta_0^{IH}(t)$. By inspection of equation (2.39), it follows that:

$$\eta_0^{IH}(t) = \frac{F_0}{\bar{\omega}^2} \tag{2.40}$$

The total solution assumes the form:

$$\eta_0(t) = a\, Sin(\bar{\omega}t + \theta) + \frac{F_0}{\bar{\omega}^2} \tag{2.41}$$

and after satisfying the initial conditions it becomes:

$$\eta_0(t) = \frac{F_0}{\bar{\omega}^2}(1 - Cos(\bar{\omega}t)) \tag{2.42}$$

Obviously, had we used the alternative approach of equation (2.31) we would have arrived at the same result.

Next we need to solve the first order equation (2.19). We begin by substituting equations (2.6) and (2.13) in equation (2.19) and equating the coefficients of the variables α_m . In this way we obtain R equations, with a typical equation having the form:

$$\ddot{\eta}_m^I(t) + \bar{\omega}^2\, \eta_m^I(t) = \tilde{f}_m^I(t) \tag{2.43}$$

The excitation term is defined as:

$$\tilde{f}_m^I(t) = 2\,\omega_m^I\, \frac{F_0}{\bar{\omega}}\, (Cos(\bar{\omega}t) - 1) \tag{2.44}$$

To obtain the solution of first order we will use this time the convolution integral approach. Since the initial conditions are zero, the total first order response is:

$$\eta_m^I(t) = 2\,\omega_m^I\, \frac{F_0}{\bar{\omega}} \int_0^t (Cos(\bar{\omega}t) - 1)\; h(t-\tau)\, d\tau \tag{2.45}$$

Carrying out the integration leads to:

$$\eta_m^I(t) = 2\,\omega_m^I\, \frac{F_0}{\bar{\omega}^3}\, (Cos(\bar{\omega}t) + \bar{\omega}t\, Sin(\bar{\omega}t) - 1) \tag{2.46}$$

As it was anticipated, equation (2.46) exhibits an unbounded behavior in time. Obviously, this is not in conformity with the physical reality: the response of an undamped oscillator subjected to a suddenly applied constant force should oscillate about the static response.

To calculate the second order solution we need to use the zeroth and first order solution to obtain each of the terms $\eta_{mn}^{II}(t)$ in the second order equations. However, if the solution given by equation (2.46) is used, the solutions of the second order equations will contain terms quadratic in t. In general, it is possible to show that if we seek a solution based on an expansion of order n, the $n-th$ term in the expansion will have a secular terms of the form t^2 [8].

The reason why the straightforward expansion fails to approximate the actual response of the system becomes apparent if we compare the total perturbation solution with the actual solution. Using equations (2.42), (2.46), (2.13) and (2.8) the approximate solution is:

$$\eta(t) \cong \frac{Fo}{\bar{\omega}^3}\,(1 - Cos(\bar{\omega}t)) + \sum_{m=1}^{R} 2\,\omega_m^I\,\frac{F_0}{\bar{\omega}^3}\,(Cos(\bar{\omega}t) + \omega t\;Sin(\bar{\omega}t) - 1)\;\alpha_m \tag{2.47}$$

whereas the exact solution is :

$$\eta(t) = \frac{F_0}{\omega(\boldsymbol{\alpha})^2}\,(1 - Cos(\omega(\boldsymbol{\alpha})t)) \tag{2.48}$$

The approximate solution (2.47) tries to reproduce the exact solution with harmonic terms that have the same undamped frequency $\bar{\omega}$ of the unperturbed system. Therefore, the approximate and exact solutions differ both in amplitude and phase. The phase difference makes the two solution stand apart more and more with time. After a certain time, the correction terms are unable to correct the unperturbed term. In fact, due to the presence of secular terms in the first order correction in equation (2.47), the fundamental assumption of the perturbation method is no longer valid.

2.2.1.4 Response Statistics of the Undamped Oscillator Subjected to a Step Force

When the response statistics are calculated with the straightforward expansion, the limitations of the method will also emerge here, as we will demonstrate next. We will consider again the undamped oscillator subjected to a step loading function, but now the parameters α_m are zero mean random variables.

First we will calculate the expected value of the response $\eta(\boldsymbol{\alpha}, t)$. As it was pointed out before, the exact closed form relationship between the response $\eta(\boldsymbol{\alpha}, t)$ and the parameters $\boldsymbol{\alpha}$ is not available. Hence, in order to apply the expected value operator to the response it is convenient to express $\eta(\boldsymbol{\alpha}, t)$ as an expansion in terms of the random variables α_m . Retaining up to first order terms we have:

$$\eta(\boldsymbol{\alpha}, t) = \eta_0(t) + \sum_{m=1}^{R} \eta_m^I(t)\,\alpha_m \tag{2.49}$$

Applying the expected value operator $E\{...\}$ to this expression we obtain:

$$E\{\eta(\boldsymbol{\alpha}, t)\} = \eta_0(t) \tag{2.50}$$

Therefore, the expected value of the response calculated with a first order straightforward expansion coincides with the response of the unperturbed system. In other words, the mean function is equal to the response of the deterministic system in which the random parameters α_m are evaluated at their mean values. If instead of equation (2.47), we use a second order expansion, we would obtain second order terms in equation (2.50). However, these terms would

have quadratic secular terms. For undamped systems the second order terms grow indefinitely and soon the expansion becomes invalid. In any case, the result obtained with the first order expansion, equation (2.50), is also wrong as it is explained next.

To demonstrate the fallacy in the solutions obtained with a straightforward expansion, we propose to carry out the following conceptual experiment. Consider a sufficiently large number of undamped oscillator with quiescent initial conditions. Let the natural frequencies of the oscillators be random variables with the same mean values. Suppose that the same deterministic loading of limited duration is applied to the oscillators at the same initial instant of time. Due to the different perturbations in the mean values of the natural frequencies of the oscillators, their responses will become more and more out of phase with time. Consider now an instant of time t' sufficiently far away from the initial instant and calculate the instantaneous mean value of the response at this instant. The initially small differences in the instantaneous values of the individual responses grew so much that at time t' they are completely random. It is logical to conclude that the sample mean of the oscillators' responses at this time approaches zero. In the limit, as we increase the number of oscillators in the experiment, the sample mean estimator tends to the mean or expected value of the response $E\{\eta(t')\}$. Evidently, the mean value calculated with a straightforward expansion of any order would never be equal to zero for the undamped case.

If we repeat the experiment with a set of damped oscillators the discrepancy in the straightforward expansion results described before will not manifest with the same intensity. In this case the individual responses of each of the oscillators would *independently tend to zero* by virtue of the damping in the system. Therefore, if we calculate the sample mean at increasing instants of time we will obtain that a sample mean that decreases with time. If we use the straightforward expansion to calculate the expected value of the response, it will yield a value that does tend to zero. However, it should be clear that this result is due to the presence of damping in the system and is not by way of probabilistic reasons as it should be.

Let us turn back our attention to the undamped oscillator subjected to a step load. We will attempt to obtain the variance of the response using a first order (straightforward) perturbation expansion. The variance of the response can be obtained from:

$$\sigma^2_{\eta(t)} = E\{(\eta(\boldsymbol{\alpha},t) - E\{\eta(\boldsymbol{\alpha},t)\})^2\} \tag{2.51}$$

Substitution of equation (2.49) in (2.51) yields:

$$\sigma^2_{\eta(t)} = 4\,\frac{F_0^2}{\bar{\omega}^6}\,(Cos(\bar{\omega}t) + \omega t\,Sin(\bar{\omega}t) - 1)^2 \sum_{m=1}^{R}\sum_{n=1}^{R} \omega_m^I \omega_n^I\, E\{\alpha_m \alpha_n\} \tag{2.52}$$

Taking into account equation (2.46) the variance becomes:

$$\sigma^2_{\eta(t)} = 4\,\frac{F_0^2}{\bar{\omega}^6}\,(Cos(\bar{\omega}t) + \bar{\omega}t\,Sin(\bar{\omega}t) - 1)^2 \sum_{m=1}^{R} \left(\omega_m^I\right)^2 \sigma_m^2 \tag{2.53}$$

If all the random variables α_m are assumed to be uncorrelated, this expression reduces to:

$$\sigma^2_{\eta(t)} = 4\,\frac{F_0^2}{\bar{\omega}^6}\,\left(Cos(\bar{\omega}t) + \omega t\,Sin(\bar{\omega}t) - 1\right)^2 \sum_{m=1}^{R} \left(\omega_m^I\right)^2 \sigma_m^2 \tag{2.54}$$

Examining the above expression we conclude that the use of a first order straightforward expansion to calculate the response variance gives rise to a quadratic secular term.

2.2.1.5 The Method of Multiple Scales

We have shown that using an approximate solution that does not take into account the change in the frequency due to the perturbations in the parameters of the oscillator gives rise to a series of problems that also affect the response statistics calculations. We will examine another approximate method that can explicitly take into account the modification of the frequency. This technique, known as the method of multiple scales, was originally conceived by Nayfeh [8] and it has been successfully used in the solution of weak nonlinear systems.

We will consider again equation (2.2) for a viscously damped oscillator in which the natural frequency is a function of R parameters $\alpha_1, \alpha_2, ..., \alpha_R$. We will again seek a solution in the form of an expansion but now the terms in the expansion will depend not only in the parameters α_m but also in products of $\alpha_m t$, $\alpha_m \alpha_n t$, etc. We can then write:

$$\eta(\boldsymbol{\alpha}, t) = \eta(\alpha_1, \alpha_2, ..., \alpha_R, \alpha_1 t, \alpha_2 t, ..., \alpha_R t, t) \tag{2.55}$$

The solution can also be written as:

$$\eta(\boldsymbol{\alpha}, t) = \eta(\alpha_1, \alpha_2, ..., \alpha_R, T_0, T_1, T_2, ..., T_R) \tag{2.56}$$

where the variables T_m are defined as:

$$T_m = \alpha_m t \tag{2.57}$$

We could have considered that the solution is also a function of terms $\alpha_m t^2$, $\alpha_m t^3$, etc. This is equivalent to keeping second, third, etc. terms in the expansions. Since the analysis with the multiple scales method is more complicated than with the straightforward expansion, in order to simplify the presentation only first order terms are retained in the expansions.

The variables T_0 and the set $T_1, T_2, ..., T_R$ can be thought of as two different time scales which justify the name of the method. Indeed, since α_m is a small perturbation, T_m represents a slower scale than the zeroth order scale $T_0 = t$.

Due to the functional relationship assumed for the response $\eta(\boldsymbol{\alpha}, t)$ in equation (2.56), the time derivatives in equation (2.2) need to be expressed in terms of the new time scales T_0 and $T_1, T_2, ...T_R$. Using the chain rule:

$$\frac{d(...)}{dt} = \frac{\partial(...)}{\partial T_0} + \sum_{m=1}^{R} \frac{\partial(...)}{\partial T_m}\alpha_m \tag{2.58}$$

$$\frac{d^2(...)}{dt^2} = \frac{\partial^2(...)}{\partial T_0^2} + 2\sum_{m=1}^{R} \frac{\partial^2(...)}{\partial T_o \partial T_m}\alpha_m \tag{2.59}$$

In order to simplify the notation in the subsequent developments we introduce two differential operators:

$$\frac{\partial(...)}{\partial T_I} = \sum_{m=1}^{R} \frac{\partial(...)}{\partial T_m}\alpha_m \tag{2.60}$$

$$\frac{\partial^2(...)}{\partial T_I^2} = 2\sum_{m=1}^{R} \frac{\partial^2(...)}{\partial T_o \partial T_m}\alpha_m \tag{2.61}$$

Using the above notation and the perturbation parameter ε as a book-keeping device to keep track the order of the magnitudes of the different terms involved, equations (2.58 - 2.59) can be written as:

$$\frac{d(...)}{dt} = \frac{\partial(...)}{\partial T_0} + \varepsilon\frac{\partial(...)}{\partial T_I} \tag{2.62}$$

$$\frac{d^2(...)}{dt^2} = \frac{\partial^2(...)}{\partial T_0^2} + \varepsilon\frac{\partial^2(...)}{\partial T_I^2} \tag{2.63}$$

It is becoming clear now that we will replace the ordinary differential equation of motion (2.2) by a partial differential equation, thus complicating the original problem. However, this apparent complication is far outweighed by the inherent advantages of the multiple scales method [8] as we will have occasion to verify it.

Alike in the straightforward expansion, the natural frequency and the loading function have to be expanded in terms of the variables α_m. The series will be truncated at the first order and we will consider that the excitation depends only on the lower time scale T_0 . Hence, $\omega(\boldsymbol{\alpha})$ and $f(\boldsymbol{\alpha}, t)$ are approximated with the expansions:

$$\omega(\boldsymbol{\alpha}) = \bar{\omega} + \sum_{m=1}^{R} \omega_m^I \alpha_m \tag{2.64}$$

$$f(\boldsymbol{\alpha}, t) = f_0(T_0) + \sum_{m=1}^{R} f_m^I(T_0)\alpha_m \tag{2.65}$$

where the superscript I indicates the first order rate of change of the functions:

$$\omega_m^I = \left.\frac{\partial \omega}{\partial \alpha_m}\right|_{\boldsymbol{\alpha}=0} \quad ; \quad f_m^I(T_0) = \left.\frac{\partial f(T_0)}{\partial \alpha_m}\right|_{\boldsymbol{\alpha}=0} \tag{2.66}$$

Introducing the following definitions:

$$\omega_1 = \sum_{m=1}^{R} \omega_m^I \, \alpha_m \tag{2.67}$$

$$f_1(T_0) = \sum_{m=1}^{R} f_m^I(T_0) \, \alpha_m \tag{2.68}$$

equations (2.64 - 2.65) can be written as:

$$\omega(\boldsymbol{\alpha}) = \bar{\omega} + \varepsilon \, \omega_1 \tag{2.69}$$

$$f(\boldsymbol{\alpha}, T_0) = f_0(T_0) + \varepsilon \, f_1(T_0) \tag{2.70}$$

The response is also expressed as a Taylor series expansion truncated at the first order:

$$\eta(\boldsymbol{\alpha}, t) = \eta_0(T_0, T_1, T_2, ..., T_R) + \sum_{m=1}^{R} \eta_m^I(T_0, T_1, T_2, ..., T_R) \, \alpha_m \tag{2.71}$$

The rates of change of the response are defined in the usual manner:

$$\eta_m^I(T_0, T_1, T_2, ..., T_R) = \left. \frac{\partial \eta(\boldsymbol{\alpha}, t)}{\partial \alpha_m} \right|_{\boldsymbol{\alpha}=0} \tag{2.72}$$

Introducing the following notation for the summation term:

$$\eta_1(T_0, T_1, T_2, ..., T_R) = \sum_{m=1}^{R} \eta_m^I(T_0, T_1, T_2, ..., T_R) \, \alpha_m \tag{2.73}$$

we can write:

$$\eta(\boldsymbol{\alpha}, t) = \eta_0(T_0, T_1, T_2, ..., T_R) + \varepsilon \, \eta_1(T_0, T_1, T_2, ..., T_R) \tag{2.74}$$

Substituting equations (2.62- 2.63), (2.69- 2.70) and (2.74) in (2.2) lead to:

$$\left(\frac{\partial^2(...)}{\partial T_0^2} + \varepsilon \, \frac{\partial^2(...)}{\partial T_I^2} \right) (\ddot{\eta}_0 + \varepsilon \, \ddot{\eta}_1) +$$

$$+ 2 \, \xi \, (\bar{\omega} + \varepsilon \, \omega_1) \left(\frac{\partial(...)}{\partial T_0} + \varepsilon \, \frac{\partial(...)}{\partial T_I} \right) (\dot{\eta}_0 + \varepsilon \, \dot{\eta}_1) +$$

$$+ (\bar{\omega} + \varepsilon \, \omega_1)^2 \, (\eta_0 + \varepsilon \, \eta_1) = f_0(T_0) + \varepsilon \, f_1(T_0) \tag{2.75}$$

Collecting the terms of the same order in the perturbation parameter ε lead to two hierarchical equations of motion:

$$O(\varepsilon^0): \ \frac{\partial^2 \eta_0}{\partial T_0^2} + 2\xi\bar{\omega}\frac{\partial \eta_0}{\partial T_0} + \bar{\omega}^2\eta_0 = f_0(T_0) \tag{2.76}$$

$$O(\varepsilon^1): \ \frac{\partial^2 \eta_1}{\partial T_0^2} + 2\xi\bar{\omega}\frac{\partial \eta_1}{\partial T_0} + \bar{\omega}^2\eta_1 = \tilde{f}_1 \tag{2.77}$$

where the excitation in the first order equation is:

$$\tilde{f}_1 = f_1(T_0) - \frac{\partial^2 \eta_0}{\partial T_1^2} - 2\xi\omega_1\frac{\partial \eta_0}{\partial T_0} - 2\bar{\omega}\omega_1\eta_0 \tag{2.78}$$

Also in this case one should bear in mind that equation (2.77) actually implies R equations of first order, each of them associated with the coefficients α_m in the definition of η_1 , equation (2.73). Moreover, it should be noticed that here also the excitation term in equation (2.78) contains the response of the equation of lower order, as well as the first two derivatives of the response.

2.2.1.6 Solution of the Zeroth Order Equation

We will proceed to examine the response of the zeroth order equation (2.76). As we discussed earlier, its solution can be written as the sum of the homogeneous and inhomogeneous solutions $\eta_0^H(t)$ and $\eta_0^{IH}(t)$, respectively:

$$\eta_0(T_0, T_1, T_2, ..., T_R) = \eta_0^H(T_0, T_1, T_2, ..., T_R) + \eta_0^{IH}(T_0) \tag{2.79}$$

The homogeneous solution has the explicit form:

$$\eta_0^H(T_0, T_1, T_2, ..., T_R) = a\, e^{-\xi\bar{\omega}T_0}\, Sin(\bar{\omega}_d T_0 + \theta) \tag{2.80}$$

It is important to note that the coefficients a and θ are constant with respect to the time scale T_0 . However, in general, they do depend on the scales T_m . That is:

$$a = a(T_1, T_2, ..., T_R) \quad ; \quad \theta = \theta(T_1, T_2, ..., T_R) \tag{2.81}$$

2.2.1.7 Solution of the First Order Equations

Substituting in equation (2.77) the operators defined in equations (2.60- 2.61) and taking into account equation (2.78) we obtain:

$$\sum_{m=1}^{R}\frac{\partial^2 \eta_m^I}{\partial T_0^2}\alpha_m + 2\xi\bar{\omega}\sum_{m=1}^{R}\frac{\partial \eta_m^I}{\partial T_0}\alpha_m + \bar{\omega}^2\sum_{m=1}^{R}\eta_m^I\alpha_m =$$

$$= \sum_{m=1}^{R} f_m^I(T_0)\,\alpha_m - 2\sum_{m=1}^{R}\frac{\partial^2 \eta_0}{\partial T_0\,\partial T_m^I}\alpha_m - 2\xi\frac{\partial \eta_0}{\partial T_0}\sum_{m=1}^{R}\omega_m^I\alpha_m - 2\bar{\omega}\eta_0\sum_{m=1}^{R}\omega_m^I\alpha_m \tag{2.82}$$

Equating coefficients of the variable α_m in both sides of the equation leads to:

$$\frac{\partial^2 \eta_m^I}{\partial T_0^2} + 2\,\xi\,\bar{\omega}\,\frac{\partial \eta_m^I}{\partial T_0} + \bar{\omega}^2\,\eta_m^I = \bar{f}_m^I \tag{2.83}$$

where:

$$\bar{f}_m^I = f_m^I(T_0) - 2\,\frac{\partial^2 \eta_0}{\partial T_0\,\partial T_m^I} - 2\,\xi\,\omega_m^I\,\frac{\partial \eta_0}{\partial T_0} - 2\,\bar{\omega}\,\omega_m^I\,\eta_0 \tag{2.84}$$

Substituting the zeroth order solution defined by equations (2.79) and (2.80) in the above equation and carrying out the derivatives indicated, equation (2.84) assumes the form:

$$\frac{\partial^2 \eta_m^I}{\partial T_0^2} + 2\,\xi\,\bar{\omega}\,\frac{\partial \eta_m^I}{\partial T_0} + \bar{\omega}^2\,\eta_m^I =$$

$$= 2\,\bar{\omega}\left(\xi\frac{\partial a}{\partial T_m} + a\,\frac{\partial \theta}{\partial T_m} + (2\xi^2 - 1)\,\omega_m^I\,a\right) e^{-\xi\bar{\omega}T_0}\,Sin(\bar{\omega}_d T_0 + \theta) +$$

$$2\,\bar{\omega}\left(\xi\,a\,\frac{\partial \theta}{\partial T_m} - \frac{\partial a}{\partial T_m} - \xi\,\omega_m^I\,a\right) e^{-\xi\bar{\omega}T_0}\,Cos(\bar{\omega}_d\,T_0 + \theta) +$$

$$+ f_m^I(T_0) - 2\,\xi\,\omega_m^I\,\frac{\partial \eta_0^{IH}(T_0)}{\partial T_0} - 2\,\bar{\omega}\,\omega_m^I\,\eta_0^{IH}(T_0) \tag{2.85}$$

The first two terms in the right hand side of equation (2.85) will give rise to secular terms in the first order solution. In order to obtain a uniform expansion, i.e. free of secular terms, the method of multiple scales impose that the coefficients of the trigonometric functions with frequency equal to the unperturbed natural frequency must vanish:

$$\xi\,\frac{\partial a}{\partial T_m} + a\,\frac{\partial \theta}{\partial T_m} + (2\xi^2 - 1)\,\omega_m^I\,a = 0 \tag{2.86}$$

$$\xi\,a\,\frac{\partial \theta}{\partial T_m} - \frac{\partial a}{\partial T_m} - \xi\,\omega_m^I\,a = 0 \tag{2.87}$$

Solving these equations we obtain the two coefficients a and θ that will guarantee that no secular terms will occur in the expansion, at least up to the first order.

2.2.1.8 Deterministic Multiple Scales Expansion for the Undamped Oscillator Subjected to a Step Force

Before proceeding with the calculations of the statistics of the response it will prove advantageous to apply the method of multiple scales to calculate the deterministic response of an oscillator subjected to an excitation described by a mathematically simple expression: a step forcing function. For simplicity, damping will be neglected and the oscillator is assumed to be initially at rest. The equation of motion reduces to equation (2.39)

$$\ddot{\eta}(\alpha, t) + \omega(\alpha)^2 \, \eta(\alpha, t) = F_0$$

The solution of order zero is:

$$\eta_0 \, (T_0, T_1, T_2, ..., T_n) = a \, Sin(\bar{\omega} t + \theta) + \frac{F_0}{\bar{\omega}^2} \tag{2.88}$$

For this particular case the equation of first order becomes:

$$\frac{\partial^2 \eta_m^I}{\partial T_0^2} + \bar{\omega}^2 \, n_m^I = 2 \bar{\omega} \left(a \frac{\partial \theta}{\partial T_m} - \omega_m^I \, a \right) Sin(\bar{\omega} T_0 + \theta) +$$

$$- 2 \bar{\omega} \frac{\partial a}{\partial T_m} Cos(\bar{\omega} T_0 + \theta) - 2 \omega_m^I \frac{F_0}{\bar{\omega}} \tag{2.89}$$

The problematic terms are the two trigonometric functions with frequency $\bar{\omega}$. Therefore, the conditions for elimination of secular terms require that:

$$a \frac{\partial \theta}{\partial T_m} - \omega_m^I \, a = 0 \tag{2.90}$$

$$\frac{\partial a}{\partial T_m} = 0 \tag{2.91}$$

From the second equation it follows that:

$$a = a_0 + O(\varepsilon^2) \tag{2.92}$$

Note that the coefficient a is constant with respect to the time scales T_m . However, it may depend on time scales of the second order (which we did not include in the analysis). This is indicated in equation (2.92) with the term $O(\varepsilon^2)$. The other coefficient θ can be found considering equations (2.90) and (2.92). It follows that:

$$\theta = \sum_{m=1}^{R} \omega_m^I \, T_m + \theta_0 + O(\varepsilon^2) \tag{2.93}$$

The solutions of the R equations of first order consist of the particular solutions only. Therefore, by inspection of equation (2.89) we can write:

$$\eta_m^I = -2\,\omega_m^I \frac{F_0}{\bar{\omega}^3} \tag{2.94}$$

Substituting the coefficients a and θ from equations (2.92- 2.93) in equation (2.80) and adding the zeroth and first order solution according to equation (2.74), we obtain:

$$\eta(t) = a_0\, Sin\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I\, \alpha_m\, t + \theta_0\right) + \frac{F_0}{\bar{\omega}^2}\left(1 - \frac{2}{\bar{\omega}} \sum_{m=1}^{R} \omega_m^I\, \alpha_m\right) \tag{2.95}$$

The final task is to obtain the undetermined coefficients a_0 and θ_0. Evaluating the above expression in $\eta(0) = \dot{\eta}(0) = 0$ it is easy to show that the total solution is:

$$\eta(t) = \frac{F_0}{\bar{\omega}^2}\left(1 - \frac{2}{\bar{\omega}} \sum_{m=1}^{R} \omega_m^I\, \alpha_m\right)\left(1 - Cos\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I\, \alpha_m\, t\right)\right) \tag{2.96}$$

It is interesting to compare the multiple scales solution, equation (2.96), with the straightforward expansion solution, equation (2.47). We immediately see that, unlike the straightforward expansion, the multiple scales solution does include a first order correction in the unperturbed natural frequency $\bar{\omega}$.

2.2.1.9 Response Statistics of the Undamped Oscillator Subjected to a Step Force

The case in which the parameters α_m are random variables with zero mean will be examined next. The objective is to calculate the mean value function and the variance function of the response $\eta(t)$.

To obtain the mean value we apply the expectation operator to equation (2.96):

$$E\{\eta(t)\} = E\left\{\frac{F_0}{\bar{\omega}^2}\left(1 - \frac{2}{\bar{\omega}} \sum_{m=1}^{R} \omega_m^I\, \alpha_m\right)\left(1 - Cos\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I\, \alpha_m\, t\right)\right)\right\} \tag{2.97}$$

This expression can be re-accommodated as follows:

$$E\{\eta(t)\} = \frac{F_0}{\bar{\omega}^2}\left(1 - E\left\{\left(1 - \frac{2}{\bar{\omega}} \sum_{m=1}^{R} \omega_m^I\, \alpha_m\right) Cos\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I\, \alpha_m\, t\right)\right\}\right) \tag{2.98}$$

Before continuing it is relevant to underscore the fact that even though we used a first order expansion, the expression for the expected value will differ from the response of the unperturbed system. This is in contrast with the results obtained with the straightforward expansion.

We need to solve the multiple integrals associated with the following expected values:

$$E\left\{Cos\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I\, \alpha_m\, t\right)\right\} =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\alpha_1\,\alpha_2\,...\,\alpha_n)\; Cos\left(\bar{\omega}t + \sum_{m=1}^{R}\omega_m^I\,\alpha_m\,t\right)\, d\alpha_1\, d\alpha_2 \dots d\alpha_R \qquad (2.99)$$

$$E\left\{\left(\sum_{m=1}^{R}\omega_m^I\,\alpha_m\right) Cos\left(\bar{\omega}t + \sum_{m=1}^{R}\omega_m^I\,\alpha_m\,t\right)\right\} =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\alpha_1\,\alpha_2\,...\,\alpha_n)\left(\sum_{m=1}^{R}\omega_m^I\,\alpha_m\right) Cos\left(\bar{\omega}t + \sum_{m=1}^{R}\omega_m^I\,\alpha_m\,t\right)\, d\alpha_1\, d\alpha_2 \dots d\alpha_R \qquad (2.100)$$

where the function $f(\alpha_1, \alpha_2, .., \alpha_n)$ is the joint probability density function of the random variables α_n . The evaluation of these integrals is rather laborious. The final results are:

$$E\left\{Cos\left(\bar{\omega}t + \sum_{m=1}^{R}\omega_m^I\,\alpha_m\,t\right)\right\} = \prod_{m=1}^{R} e^{-\frac{(\sigma_m\omega_m^I t)^2}{2}}\, Cos(\bar{\omega}t) \qquad (2.101)$$

$$E\left\{\left(\sum_{m=1}^{R}\omega_m^I\,\alpha_m\right) Cos\left(\bar{\omega}t + \sum_{m=1}^{R}\omega_m^I\,\alpha_m\,t\right)\right\} =$$

$$= -t\, Sin(\bar{\omega}t)\sum_{m=1}^{R}\left(\left(\sigma_m\omega_m^I\right)^2 e^{-\frac{(\sigma_m\omega_m^I t)^2}{2}} \prod_{n=1;n\neq m}^{R} e^{-\frac{(\sigma_m\omega_m^I t)^2}{2}}\right) \qquad (2.102)$$

Therefore, the expected value of the response can be written as:

$$E\{\eta(t)\} = \frac{F_0}{\bar{\omega}^2}\left(1 - \left(Cos(\bar{\omega}t) + Q(t)\, Sin(\bar{\omega}t)\right) R(t)\right) \qquad (2.103)$$

where the following notation is used:

$$P = \sum_{m=1}^{R}\left(\sigma_m\omega_m^I\right)^2 \qquad (2.104)$$

$$Q(t) = \frac{2}{\bar{\omega}} P\, t \qquad (2.105)$$

$$R(t) = e^{-\frac{1}{2}P\,t^2} \qquad (2.106)$$

Next we will determine the variance of the response. We begin by calculating first the mean square value of the response. Using equation (2.96), we can write:

$$\eta(t)^2 = \left(\frac{F_0}{\bar{\omega}^2}\right)^2 \left(1 - \frac{4}{\bar{\omega}} \sum_{m=1}^{R} \omega_m^I \, \alpha_m + \right.$$

$$+ 2\left(\frac{4}{\bar{\omega}} \sum_{m=1}^{R} \omega_m^I \, \alpha_m - 1\right) Cos\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I \, \alpha_m \, t\right) -$$

$$\left. - \left(\frac{4}{\bar{\omega}} \sum_{m=1}^{R} \omega_m^I \, \alpha_m - 1\right) Cos^2\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I \, \alpha_m \, t\right)\right) \tag{2.107}$$

The double summation terms in the variables $\alpha_m \alpha_n$ have been neglected in the above equation.

Applying the expected value operator on equation (2.107) and recalling that the variables α_m have zero mean, we obtain:

$$E\{\eta(t)^2\} = \left(\frac{F_0}{\bar{\omega}^2}\right)^2 \left(1 + 2\,E\left\{\left(\frac{4}{\bar{\omega}} \sum_{m=1}^{R} \omega_m^I \, \alpha_m - 1\right) Cos\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I \, \alpha_m \, t\right)\right\} - \right.$$

$$\left. - \; E\left\{\left(\frac{4}{\bar{\omega}} \sum_{m=1}^{R} \omega_m^I \, \alpha_m - 1\right) Cos^2\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I \, \alpha_m \, t\right)\right\}\right) \tag{2.108}$$

To proceed further we need to calculate the expected values of expressions that involve the square of the cosine function. This, in turn, requires to evaluate the following multiple integrals:

$$E\left\{Cos^2\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I \, \alpha_m \, t\right)\right\} =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\alpha_1 \, \alpha_2 \ldots \alpha_R) \; Cos^2\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I \, \alpha_m \, t\right) \, d\alpha_1 \, d\alpha_2 \ldots d\alpha_R \tag{2.109}$$

$$E\left\{\left(\sum_{m=1}^{R} \omega_m^I \, \alpha_m\right) Cos^2\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I \, \alpha_m \, t\right)\right\} =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\alpha_1 \, \alpha_2 \ldots \alpha_R) \left(\sum_{m=1}^{R} \omega_m^I \, \alpha_m\right) Cos^2\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I \, \alpha_m \, t\right) d\alpha_1 \, d\alpha_2 \ldots d\alpha_R \tag{2.110}$$

After performing the evaluation of the integrals we obtain:

$$E\left\{Cos^2\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I\, \alpha_m\, t\right)\right\} = \frac{1}{2}\left(1 + \prod_{m=1}^{R} e^{-2(\sigma_m \omega_m^I t)^2}\, Cos\,(2\bar{\omega}t)\right) \tag{2.111}$$

$$E\left\{\left(\sum_{m=1}^{R} \omega_m^I\, \alpha_m\right) Cos^2\left(\bar{\omega}t + \sum_{m=1}^{R} \omega_m^I\, \alpha_m\, t\right)\right\} =$$

$$= -t\, Sin\,(2\bar{\omega}t) \sum_{m=1}^{R} \left(\left(\sigma_m \omega_m^I\right)^2 e^{-2(\sigma_m \omega_m^I t)^2} \prod_{n=1;n\neq m}^{R} e^{-2(\sigma_n \omega_n^I t)^2}\right) \tag{2.112}$$

Using the above results along with the expected values of the expressions in equation (2.101-2.102) and (2.104- 2.106), the mean square response can be written as:

$$E\{\eta(t)^2\} = \left(\frac{F_0}{\bar{\omega}^2}\right)^2 \left(\frac{3}{2} - 2\,(Cos\,(\bar{\omega}t) + Q(t)\, Sin\,(\bar{\omega}t))\, e^{-\frac{1}{2}P\,t^2} + \right.$$

$$\left. + \frac{1}{2}(Cos\,(2\bar{\omega}t) + 4Q(t)\, Sin\,(2\bar{\omega}t))\, e^{-2\,P\,t^2}\right) \tag{2.113}$$

We note that as time passes, the mean square response tends to the value of the unperturbed system. That is,

$$\lim_{t\to\infty} E\left\{\eta(t)^2\right\} = \frac{3}{2}\left(\frac{F_0}{\bar{\omega}^2}\right)^2 \tag{2.114}$$

Finally, to obtain the response variance function we have to substitute equations (2.103) and (2.113) in the following expression:

$$\sigma^2_{\eta(t)} = E\{\eta(t)^2\} - (E\{\eta(t)\})^2 \tag{2.115}$$

2.3 Multi-Degree-of-Freedom Systems

We will consider structural systems in which the mass and stiffness matrices as well as the excitation vector are function of the set of parameters $\alpha_1, \alpha_2, ..., \alpha_R$. These variables represent perturbations on the values of a certain set of parameters of the system. We are specially interested in the case in which the parameters α_m represent random variations about the mean values. Using a vector $\boldsymbol{\alpha}$ to collect the set of random variables, the equations of motion can be written as:

$$[M(\boldsymbol{\alpha})]\; \ddot{\boldsymbol{u}}(\boldsymbol{\alpha},t) + [C]\; \dot{\boldsymbol{u}}(\boldsymbol{\alpha},t) + [K(\boldsymbol{\alpha})]\; \boldsymbol{u}(\boldsymbol{\alpha},t) = \boldsymbol{F}(\boldsymbol{\alpha},t) \tag{2.116}$$

The excitation vector is assumed to be a function of the parameters α_m in order to include excitations of seismic origin in the analysis. In this case the vector $\boldsymbol{F}(\boldsymbol{\alpha},t)$ assumes the form:

$$\boldsymbol{F}(\boldsymbol{\alpha},t) = -[M(\boldsymbol{\alpha})]\,\mathbf{r}\,\ddot{x}_g(t) \tag{2.117}$$

where $\ddot{x}_g(t)$ is the acceleration of the base and $\mathbf{r}$ is the vector of influence coefficients.

The solution of the equations of motion (2.116) can be obtained applying perturbation techniques to the coupled equations of motion or to the uncoupled system of equations. We will examine both cases.

2.3.1 Perturbation Method Applied to the Coupled Equations of Motion

To obtain a solution in the form of a perturbation expansion it is necessary to replace all the system matrices as well as the response and excitation vectors by their respective Taylor series expansion with respect to the variables $\boldsymbol{\alpha}$. Keeping only first order terms, we can write:

$$[K(\boldsymbol{\alpha})] = [\bar{K}] + \sum_{m=1}^{R} \left[K_m^I\right] \alpha_m \tag{2.118}$$

$$[M(\boldsymbol{\alpha})] = [\bar{M}] + \sum_{m=1}^{R} \left[M_m^I\right] \alpha_m \tag{2.119}$$

$$\boldsymbol{F}(\boldsymbol{\alpha},t) = \bar{\boldsymbol{F}}(t) + \sum_{m=1}^{R} \boldsymbol{F}_m^I(t)\,\alpha_m \tag{2.120}$$

$$\boldsymbol{u}(\boldsymbol{\alpha},t) = \boldsymbol{u}_0(t) + \sum_{m=1}^{R} \boldsymbol{u}_m^I(t)\,\alpha_m \tag{2.121}$$

To shorten the notation the following terms are defined:

$$[K_1] = \sum_{m=1}^{R} \left[K_m^I\right] \alpha_m \tag{2.122}$$

$$[M_1] = \sum_{m=1}^{R} \left[M_m^I\right] \alpha_m \tag{2.123}$$

$$\boldsymbol{F}_1(t) = \sum_{m=1}^{R} \boldsymbol{F}_m^I(t)\,\alpha_m \tag{2.124}$$

$$\boldsymbol{u}_1(t) = \sum_{m=1}^{R} \boldsymbol{u}_m^I(t)\,\alpha_m \tag{2.125}$$

Equations (2.118- 2.121) can then be expressed as:

$$[K(\boldsymbol{\alpha})] = [\bar{K}] + \varepsilon\,[K_1] \tag{2.126}$$

$$[M(\boldsymbol{\alpha})] = [\bar{M}] + \varepsilon\,[M_1] \tag{2.127}$$

$$\boldsymbol{F}(\boldsymbol{\alpha}, t) = \bar{\boldsymbol{F}}(t) + \varepsilon \boldsymbol{F}_1(t) \tag{2.128}$$

$$\boldsymbol{u}(\boldsymbol{\alpha}, t) = \boldsymbol{u}_0(t) + \varepsilon \boldsymbol{u}_1(t) \tag{2.129}$$

Substituting the expansions in equations (2.126- 2.129) in the equations of motion (2.116) gives:

$$\begin{aligned} &([\bar{M}] + \varepsilon [M_1])(\ddot{\boldsymbol{u}}(t) + \varepsilon \ddot{\boldsymbol{u}}_1(t)) + [C](\dot{\boldsymbol{u}}_0(t) + \varepsilon \dot{\boldsymbol{u}}_1(t)) + \\ &+ ([\bar{K}] + \varepsilon [K_1])(\boldsymbol{u}_0(t) + \varepsilon \boldsymbol{u}_1(t)) = \bar{\boldsymbol{F}}(t) + \varepsilon \boldsymbol{F}_1(t) \end{aligned} \tag{2.130}$$

The following hierarchical equations are obtained after collecting terms of the same order in the perturbation parameter ε:

$$O(\varepsilon^o) = [\bar{M}] \ddot{\boldsymbol{u}}_0(t) + [C] \dot{\boldsymbol{u}}_0(t) + [\bar{K}] \boldsymbol{u}_0(t) = \bar{\boldsymbol{F}}(t) \tag{2.131}$$

$$O(\varepsilon^1) = [\bar{M}] \ddot{\boldsymbol{u}}_1(t) + [C] \dot{\boldsymbol{u}}_1(t) + [\bar{K}] \boldsymbol{u}_1(t) = \bar{\boldsymbol{F}}_1(t) \tag{2.132}$$

where:

$$\bar{\boldsymbol{F}}_1(t) = \boldsymbol{F}_1(t) - [M_1] \ddot{\boldsymbol{u}}_0(t) - [K_1] \boldsymbol{u}_0(t) \tag{2.133}$$

It is important to keep in mind that equation (2.133) actually implies R systems of equations. This is so because the vectors $\boldsymbol{u}(t)$ and $\boldsymbol{F}(t)$ are defined as linear combinations of R vectors $u_m^I(t)$ and $F_m^I(t)$, respectively. Note also that the two sets of equations (2.131- 2.132) possess the same matrix differential operator and only differ in the excitation term.

It will be assumed that the damping matrix corresponds to a classical damping model or that the system has Rayleigh's damping. Hence, the zeroth and higher order systems can be decoupled using the following transformations:

$$\boldsymbol{u}_0(t) = [\bar{\Phi}] \boldsymbol{\eta}_0(t) \tag{2.134}$$

$$\boldsymbol{u}_1(t) = [\bar{\Phi}] \boldsymbol{\eta}_1(t) \tag{2.135}$$

The eigenvectors in $[\bar{\Phi}]$ are those of the undamped unperturbed system. They are obtained from the solution of the eigenproblem:

$$\{[\bar{K}] - \bar{\lambda}_i [\bar{M}]\} \bar{\boldsymbol{\phi}}_i = \boldsymbol{0} \tag{2.136}$$

Substituting the expressions (2.134- 2.135) in equations (2.131- 2.132) and premultiplying on the

right by the transpose of the matrix $[\bar{\Phi}]$ leads to the following two sets of uncoupled differential equations:

$$O(\varepsilon^0): \quad \ddot{\eta}_{i_0}(t) + 2\,\xi_i\,\bar{\omega}_i\,\dot{\eta}_{i_0}(t) + \bar{\omega}_i^2\,\eta_{i_0}(t) = N_{i_0}(t) \tag{2.137}$$

$$O(\varepsilon^1): \quad \ddot{\eta}_{i_1}(t) + 2\,\xi_i\,\bar{\omega}_i\,\dot{\eta}_{i_1}(t) + \bar{\omega}_i^2\,\eta_{i_1}(t) = \tilde{N}_{i_1}(t) \tag{2.138}$$

where the modal excitation terms are given by:

$$N_{i_0}(t) = \bar{\boldsymbol{\phi}}_i^T \bar{\boldsymbol{F}}(t) \tag{2.139}$$

$$\tilde{N}_{i_1}(t) = \bar{\boldsymbol{\phi}}_i^T \boldsymbol{F}_1(t) - \boldsymbol{\Gamma}_i^T\,\ddot{\boldsymbol{\eta}}_0(t) - \boldsymbol{\Psi}_i^T\,\boldsymbol{\eta}_0(t) \tag{2.140}$$

The vectors $\boldsymbol{\Gamma}_i$ and $\boldsymbol{\Psi}_i$ in the last equation are defined as:

$$\boldsymbol{\Gamma}_i^T = \bar{\boldsymbol{\phi}}_i^T\,[M_1]\,[\bar{\Phi}] \tag{2.141}$$

$$\boldsymbol{\Psi}_i^T = \bar{\boldsymbol{\phi}}_i^T\,[K_1]\,[\bar{\Phi}] \tag{2.142}$$

Recalling that the vector $\boldsymbol{u}_1(t)$ is defined as an expansion in terms of the parameters α_m and that $\boldsymbol{\eta}_1(t)$ and $\boldsymbol{u}_1(t)$ are related through the transformation in equation (2.135), we can express $\boldsymbol{\eta}_1(t)$ as:

$$\boldsymbol{\eta}_1(t) = \sum_{m=1}^{R} \boldsymbol{\eta}_m^I\,\alpha_m \tag{2.143}$$

Each of the first order terms identified with the subscript 1 in equation (2.138) is defined as a summation of derivatives multiplied by the variables α_m . Substituting these expressions and equating the coefficients of the same variable α_m one obtains R equations with the form:

$$\ddot{\eta}_{i_m}^I(t) + 2\,\xi_i\,\bar{\omega}_i\,\dot{\eta}_{i\,m}^I(t) + \bar{\omega}_i^2\,\eta_{i_m}^I(t) =$$

$$= \sum_{m=1}^{N} \bar{\Phi}_{ji}\,\boldsymbol{F}_{j_m}^I(t) - \Gamma_{ii_m}\,\ddot{\eta}_{i_0}(t) - \sum_{k=1;k\neq i}^{N} \Gamma_{ik_m}\,\ddot{\eta}_{k_0}(t) -$$

$$- \Psi_{ii_m}\,\eta_{i_0}(t) - \sum_{k=1;k\neq i}^{N} \Psi_{ik_m}\,\eta_{k_0}(t) \tag{2.144}$$

in which $\boldsymbol{F}_{j_m}^I$ and Γ_{ik_m} are the elements of the following two vectors:

$$\boldsymbol{\Gamma}_{i_m}^T = \bar{\phi}_i^T \left[M_m^I\right] [\bar{\Phi}] \tag{2.145}$$

$$\Psi_{i_m}^T = \bar{\phi}_i^T \left[K_m^I\right] [\bar{\Phi}] \tag{2.146}$$

It is seen that the excitation in the $i-th$ equation of first order includes the $i-th$ modal acceleration $\ddot{\eta}_{i_0}(t)$ and displacement $\eta_{i_0}(t)$ obtained from the solution of the zeroth order equation.

These functions will act as resonant loading for the first order equations, therefore introducing secular terms in the solutions. Liu, Besterfield and Belytschko [39,40] proposed to use a procedure based on a numerical Fourier transform to filter out the harmonic components in the excitation of the first order equation. Eliminating in this way the terms that lead to secular terms they were able to obtain a uniform expansion. It should be pointed out that the secular terms appear because the solution is only valid for short periods of time. Although by eliminating these terms, the expansion is rendered uniform, this process does not intrinsically improve the approximate solution. Indeed, the method still tries to estimate the exact response using terms with frequencies equal to the natural frequencies of the unperturbed system.

2.3.2 Perturbation Method Applied to the Decoupled Equations of Motion

In this section we will present an alternative way to apply the perturbation technique for the solution of the equations of motion (2.116). Although we may not know this matrix, it is always possible to decouple the equations of motion using the eigenvector matrix $[\Phi(\alpha)]$ in the transformation:

$$u(\alpha, t) = [\Phi(\alpha)]\, \eta(\alpha, t) \tag{2.147}$$

The only conditions are those usual in modal analysis, namely that the system must be linear and classically or proportionally damped. If the system has Rayleigh damping, that is if the matrix $[C]$ is defined as a linear combination of $[K(\alpha)]$ and $[M(\alpha)]$, the equations of motion can also be decoupled, but the modal damping ratios would be function of the parameters α_m . We are not going to study this case and we will concentrate in the case in which the damping ratios ξ_i are all independent of α_m . The uncoupled equations of motion obtained from the transformation (2.147) are:

$$\ddot{\eta}_i(\alpha, t) + 2\,\xi_i\,\omega_i(\alpha)\,\dot{\eta}_i(\alpha, t) + \omega_i(\alpha)^2\,\eta_i(\alpha, t) = N_i(\alpha, t) \tag{2.148}$$

where the i-th generalized force is:

$$N_i(\alpha, t) = \phi_i(\alpha)^T\, F(\alpha, t) \tag{2.149}$$

This method is more amenable for analytical study using the results of Section 2.2.2. Therefore, it is adopted to examine in detail a specific example.

2.3.2.1 Deterministic Response of Multi Dof System Subjected to a Step Force

Consider an undamped multi-dof system in which the excitation is defined by a mathematically simple expression. The same step loading function previously used for single dof system is selected. The excitation vector is then:

$$\boldsymbol{F}(t) = F_0 \, \boldsymbol{S} \, U(t) \tag{2.150}$$

where $\boldsymbol{S}$ is a vector with constant coefficients, F_0 is the magnitude of the load and $U(t)$ is the Heaviside function. Ignoring the system damping the equation (2.148) reduces to:

$$\ddot{\eta}_i(\boldsymbol{\alpha}, t) + \omega_i(\boldsymbol{\alpha})^2 \, \eta_i(\boldsymbol{\alpha}, t) = N_i(\boldsymbol{\alpha}, t) \tag{2.151}$$

where:

$$N_i(\boldsymbol{\alpha}, t) = F_0 \, \boldsymbol{\phi}_i(\alpha)^T \, \boldsymbol{S} \, U(t) \tag{2.152}$$

A crucial simplification will be made at this point. We will neglect the effect of the parameters α_m on the eigenvectors $\boldsymbol{\phi}(\boldsymbol{\alpha})$ in the definition of the modal excitation N_i . A similar assumption was used by Prasthofer and Beadle [11] in their study of the response of a system with a single random parameter. This assumption will greatly simplify the calculations of the response statistics in the next section. We can then write:

$$N_i(\boldsymbol{\alpha}, t) \simeq \bar{N}_i(t) = F_0 \, \bar{\boldsymbol{\phi}}_i^T \boldsymbol{S} U(t) \tag{2.153}$$

Recalling the expression (2.96) in Section 2.2.2 for the response of a single dof system subjected to a step force we have that:

$$\eta_i(t) = \frac{Ni}{\bar{\omega}_i^2} \left(1 - \frac{2}{\bar{\omega}_i} \sum_{m=1}^{R} \omega_{i_m}^I \, \alpha_m \right) \left(1 - Cos \left(\bar{\omega}_i t + \sum_{m=1}^{R} \omega_{i_m}^I \, \alpha_m \, t \right) \right) \tag{2.154}$$

where

$$N_i = F_0 \, \bar{\boldsymbol{\phi}}_i^T \boldsymbol{S}$$

To obtain the response in terms of the physical coordinates $\boldsymbol{u}(t)$ we will make use of the same simplification as in equation (2.153):

$$u_i(t) = \sum_{j=1}^{N} \bar{\Phi}_{ij} \, \eta_j(t) \tag{2.155}$$

2.3.2.2 Response Statistics of a Multi-Dof System Subjected to a Step Force

We will consider now the problem of determining the statistics of the response of a multi-dof system in which the mass and stiffness coefficients are function of random variables $\alpha_1, \alpha_2, ..., \alpha_R$. As before, these variables represent random perturbation about the mean values. The mean values will be assumed equal to zero.

The mean value of the displacement of the $i-th$ dof can be calculated with equation (2.155):

$$E\{u_i(t)\} = \sum_{j=1}^{N} \bar{\Phi}_{ij}\, E\{\eta_j(t)\} \tag{2.156}$$

Using the results of Section 2.2.2 the expected value of the modal displacement $\eta_j(t)$ is defined as:

$$E\{\eta_j(t)\} = \frac{N_j}{\bar{\omega}_j^2}\left(1 - (Cos(\bar{\omega}_j t) + Q_j(t)\, Sin(\bar{\omega}_j t))\, e^{-\frac{1}{2} P_j t^2}\right) \tag{2.157}$$

where:

$$P_j = \sum_{m=1}^{R} \left(\sigma_m\, \omega_{jm}^{I}\right)^2 \tag{2.158}$$

$$Q_j(t) = \frac{2}{\omega_j} P_j\, t \tag{2.159}$$

To obtain the variance of the response vector $u(t)$ we proceed as follows. The variance of the $i-th$ displacement can be calculated according to:

$$\sigma_{ui(t)}^2 = E\{u_i(t)^2\} - (E\{u_i(t)\})^2 \tag{2.160}$$

Recalling equation (2.156) the second term in the right hand side of the previous equation can be written as:

$$(E\{u_i(t)\})^2 = \sum_{j=1}^{N}\sum_{k=1}^{N} \bar{\Phi}_{ij}\, \bar{\Phi}_{ik}\, E\{\eta_j(t)\}\, E\{\eta_k(t)\} \tag{2.161}$$

Moreover, according to equation (2.155):

$$u_i(t)^2 = \sum_{j=1}^{N}\sum_{k=1}^{N} \bar{\Phi}_{ij}\, \bar{\Phi}_{ik}\, \eta_j(t)\, \eta_k(t) \tag{2.162}$$

It is convenient to write this expression as:

$$u_i(t)^2 = \sum_{j=1}^{N} \bar{\Phi}_{ij}^2\, \eta_j(t)^2 + 2 \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \bar{\Phi}_{ij}\, \bar{\Phi}_{ik}\, \eta_j(t)\, \eta_k(t) \tag{2.163}$$

The first term in equation (2.160) can be obtained calculating the expected value of the above expression:

$$E\{u_i(t)^2\} = \sum_{j=1}^{N} \bar{\Phi}_{ij}^2\, E\{\eta_j(t)^2\} + 2 \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \bar{\Phi}_{ij}\, \bar{\Phi}_{ik}\, E\{\eta_j(t)\, \eta_k(t)\} \tag{2.164}$$

The first term of this equation is the mean square value of the $j-th$ modal displacement. This can be calculated using equation (2.113) for a single dof system:

$$E\{\eta_j(t)^2\} = \left(\frac{N_j}{\bar{\omega}_j^2}\right)^2 \left(\frac{3}{2} - \right.$$

$$- 2\left(Cos\left(\bar{\omega}_j t\right) + 2\,Q_j(t)\,Sin\left(\bar{\omega}_j t\right)\right) e^{-\frac{1}{2}P_j t^2} +$$

$$\left. + \frac{1}{2}\left(Cos\left(2\bar{\omega}_j t\right) + 4\,Q_j(t)\,Sin\left(2\bar{\omega}_j t\right)\right) e^{-2P_j t^2}\right) \tag{2.165}$$

where P_j and $Q_j(t)$ are defined in equations (2.158 and 2.159).

The terms in the second summation in the mean square value of the response are the expected values of the product $\eta_j(t)\eta_k(t)$. After some algebraic manipulations, the expression for these expected values reduces to:

$$E\{\eta_j(t)\,\eta_k(t)\} = \left(\frac{N_j}{\bar{\omega}_j^2}\frac{N_k}{\bar{\omega}_k^2}\right)(1 -$$

$$- A_{jk}(t)\,e^{-\frac{1}{2}Pj\,t^2} - A_{kj}(t)\,e^{-\frac{1}{2}Pk\,t^2} +$$

$$+ B_{jk}(t)\,e^{-\frac{1}{2}S_{jk}t^2} + C_{jk}(t)\,e^{-\frac{1}{2}D_{jk}t^2} \tag{2.166}$$

where the following definitions have been introduced:

$$A_{rs}(t) = Cos\left(\bar{\omega}rt\right) + 2t\left(\frac{P_r}{\bar{\omega}r} + \frac{Q_{rs}}{\bar{\omega}s}\right) Sin\left(\bar{\omega}_r t\right) \tag{2.167}$$

$$B_{rs}(t) = \frac{1}{2}\left(Cos\left(\left(\bar{\omega}_r + \bar{\omega}_s\right)t\right) + 2t\left(\frac{M_{rs}}{\bar{\omega}r} + \frac{M_{sr}}{\bar{\omega}s}\right) Sin\left(\left(\bar{\omega}_r + \bar{\omega}_s\right)t\right)\right) \tag{2.168}$$

$$C_{rs}(t) = \frac{1}{2}\left(Cos\left(\left(\bar{\omega}_r + \bar{\omega}_s\right)t\right) + 2t\left(\frac{N_{rs}}{\bar{\omega}r} + \frac{N_{sr}}{\bar{\omega}s}\right) Sin\left(\left(\bar{\omega}_r + \bar{\omega}_s\right)t\right)\right) \tag{2.169}$$

$$Q_{rs} = \sum_{m=1}^{R} \sigma_m^2\,\omega_{rm}^I\,\omega_{sm}^I \tag{2.170}$$

$$M_{rs} = \sum_{m=1}^{R} \sigma_m^2\,\omega_{rm}^I\left(\omega_{rm}^I + \omega_{sm}^I\right) \tag{2.171}$$

$$N_{rs} = \sum_{m=1}^{R} \sigma_m^2 \, \omega_{r_m}^I \, \left(\omega_{r_m}^I - \omega_{s_m}^I\right) \tag{2.172}$$

$$S_{rs} = \sum_{m=1}^{R} \sigma_m^2 \, \left(\omega_{r_m}^I + \omega_{s_m}^I\right)^2 \tag{2.173}$$

$$D_{rs} = \sum_{m=1}^{R} \sigma_m^2 \, \left(\omega_{r_m}^I - \omega_{s_m}^I\right)^2 \tag{2.174}$$

Finally, the variance of the response is calculated from equation (2.160) with the two terms defined by equations (2.161) and (2.164) along with equations (2.165) and (2.166).

2.3.2.3 Numerical Examples

The plane frame shown in Figure (1.1) will be used to present the calculation of time-history response statistics for a multi-dof system.

We will examine the undamped response to a step loading function acting on joint (a) in the horizontal direction. The stiffness coefficients of the connections at the beam's ends are considered as uncorrelated normal random variables. The mean value of the stiffness coefficients *is* such that the corresponding fixity factor is 0.50, and two different values of c.o.v., 0.10 and 0.20, will be used to describe the degree of uncert*ainty in* the distribution.

Figure (2.1) shows the time history for the expected value of the horizontal displacement of joint displacement of joint (a). The curves were normalized by dividing them by the maximum value of displacement of the system with (deterministic) rigid joints. The time ordinates were also divided by the fundamental natural period of the rigid system. The effect of the uncertainty associated to the connection stiffnesses can be seen in this figure: instead of oscillating with constant amplitude, the expected value of the displacement presents a decrement for successive cycles. As time increases, the expected displacement tends to a constant value. This effect can be also seen in Figure (2.2) which shows the expected value and the mean square value of the horizontal displacement of joint (a), for a c.o.v. equal to 0.20. As time increases, the mean square value also tends to a constant value.

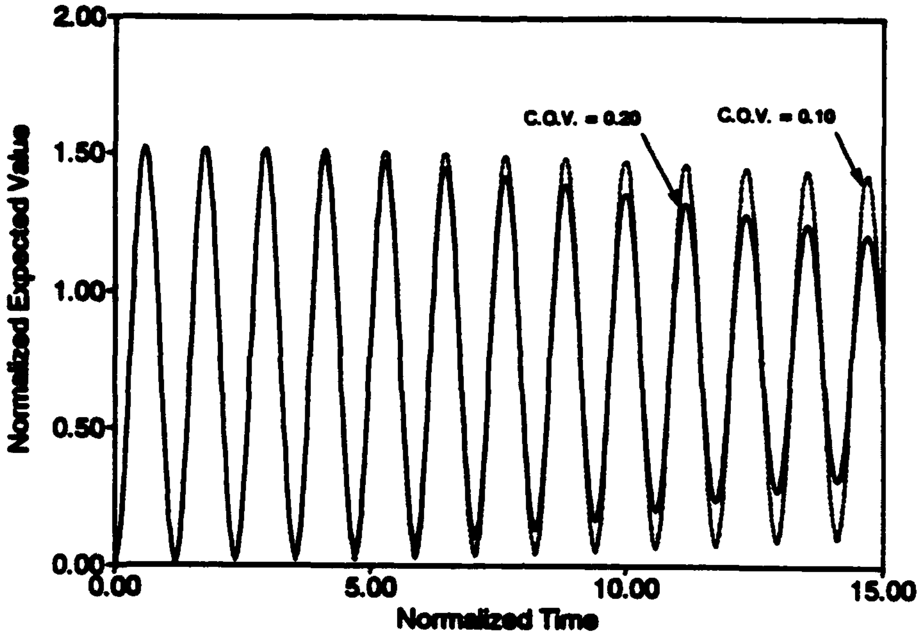

Figure 2.1: Expected Value: Horizontal Displacement of Joint (a).

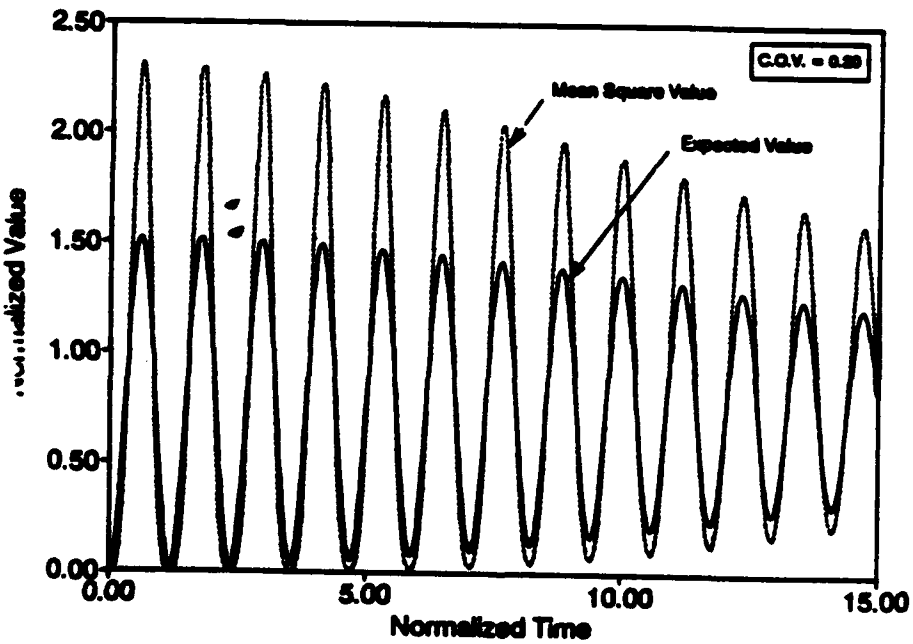

Figure 2.2: Expected Value and Mean Square Value: Horizontal Displacement of Joint (a).

Chapter 3

Conclusions and Recommendations for Future Work

3.1 Conclusions

A finite element model of a beam element was formulated to incorporate the effects of the flexibility and eccentricity of the end connections on the dynamic behavior. A variational formulation was employed to obtain the mass and stiffness matrices that consider both effects. The system matrices were expressed as the sum of standard finite element matrices plus correction matrices.

The model developed was used in several numerical examples to examine how the dynamic properties and response of structural elements are affected by the finite size and flexibility of the connections. It was observed that the lower natural frequencies diminish as the connection stiffness reduces, whereas their values increase with the joint size. It was shown that a structure with initially rigid joints and non-coincident natural frequencies, may present frequency crossing phenomena when the stiffness of the joints is changed. For structures subjected to localized harmonic excitations it was shown that changes in connection stiffness can alter the frequency which presents the maximum amplification. The case of structures subjected to seismic excitations was also considered. Several design response quantities of a ten-story framework were calculated using a response spectrum method and a modified SRSS rule to combine the modal responses. It was shown that the joint characteristics (stiffness and size) have a more pronounced effect on the design displacements than on the design forces. For the structure considered, it was observed that an increase in the size cf the connections provokes an increase in the design base shear and a decrease in the design displacements.

Expressions to calculate the derivatives of eigenvalues and eigenvectors with respect to the fixity factors were derived explicitly, and they were used to study the eigenvalue sensitivity to changes in the stiffness of the joints. It was shown that, for a given structure and depending on which joints are perturbed, there is a maximum sensitivity interval for each eigenvalue. It was also observed that it is possible to determine a certain pattern of perturbations that will produce the maximum variation of a given eigenvalue. In general, this is achieved by introducing on selected joints, perturbations which do not have necessarily the same magnitude.

To take into account the uncertainty in the values of the connection stiffness, they were

modelled as random variables. A second order perturbation approach was used to formulate a stochastic finite element model, which was used to predict the statistics of the eigenvalues and eigenvectors. The results were validated by comparing them against a Monte-Carlo Simulation. An excellent agreement was observed. The numerical examples showed that the level of dispersion induced by the uncertainty of the connection stiffness varies with the eigenvalue number and is different for the different components of the eigenvectors. For the example considered, it was found that an almost linear statistical relation exists between the eigenproperties and the stiffness of the joints.

The calculation of the statistics of the time-history response of single and multi-dof structures with random parameters was studied. It was shown that the use of the straightforward expansion leads to results that are not in agreement with the physical reality, from a probabilistic point of view. A solution methodology based on the method of multiple scales was proposed to overcome the shortcomings associated with the straightforward expansion technique. The formulation developed was used to determine the response statistics of a structure subjected to a simple loading case.

3.2 Recommendations

It is hoped that the work in these two reports can open the door for future studies that will further enhance the understanding of the phenomenon. Some of the possible extensions and generalizations of this study are the following:

The combined effect of the connections flexibility associated with rotational, torsional and extensional degrees of freedom can be included in the finite element model.

The energy dissipation in the joints can be taken into account by means of rotational dampers at the joints. An equivalent viscous damping model can be used to define the coefficients of the dampers. The corresponding damping matrix may be developed following the methodology presented in Chapter 2 of the first report.

The sensitivity analysis presented in Chapter 3 of the first report can be also be extended to include the effect of localized damping in the joints.

The effect of the correlation among the random variables representing the stiffness of the connections can be included in the stochastic eigenvalue problem in Chapter 1 of the second report.

The methodology developed in Chapter 2 of the second report can be generalized to include more complicated loading functions than those used in this study. For instance, the case of a linearly varying excitation can be very useful to study the response due to loadings defined by time histories formed by linear segments, such as earthquake accelerograms. In this case one should also consider the case in which the initial conditions are different from zero at the beginning of the time interval in which the load is divided.

The effect of damping, neglected in the analysis in Chapter 2 of Part II, can be taken into

account in the calculation of the response statistics. This is specially important to be able to treat seismic excitations or other non-impulsive loads. The modal damping ratios can be regarded as deterministic variables or, in a more general but also more complicated case, as random variables.

Finally, the nonlinear characteristics of the moment-rotation curves should be taken into account. In this case the finite element matrices developed in Chapter 2 of Part I can be used as "tangent matrices" for the nonlinear analysis.

Bibliography

[1] Kato, B., "Beam-to-Beam Column Connection Research in Japan", Journal of the Structural Division, ASCE, 108 (ST2), 343-360, 1982.

[2] Krawinkler, H. and Popov, E. P., "Seismic Behavior of Moment Resisting Connections and Joints", Journal of the Structural Division, ASCE, 108 (ST2), 373-391, 1982.

[3] Altman, W.G., Azizinamini, A., Bradburn, J.H. and Radziminski, J.B., "Moment-Rotation Characteristics of Semi-Rigid Steel Beam-Column Connections", Structural Research Studies, Department of Civil Engineering, University of South Carolina, Columbia, SC, June, 1982.

[4] Liu, W.K., Besterfield, G. and Belytschko T., "Variational Formulation for Probabilistic Mechanics", Finite Elements Methods for Plates and Shells Structures: Formulation and Algorithms, eds. T.J.R. Hughes and E. Hinton, Pineridge Press, Swansea, England, 1986.

[5] Liu, W.K., Besterfield, G. and Belytschko T., "Transient Probabilistic Systems", Computer Methods in Applied Mechanics and Engineering, 67, 27-54, 1988.

[6] STAT/LIBRARY: Fortran Subroutines for Statistical Analysis, Version 1.0, User's Manual, IMSL, Inc., Houston, TX, 1987.

[7] Suarez, L.E., and Matheu, E.E., "Joint Flexibility Effects on the Dynamic Response of Structures - Part I: Deterministic Analysis", Technical Report, College of Engineering, University of Puerto Rico - Mayagüez, 1992.

[8] Nayfeh, A.H., Introduction to Perturbation Techniques, John Wiley & Sons, New York, NY, 1981.

[9] Prasthofer, P.H., and Beadle, C., "Dynamic Response of Struc- tures with Statistical Uncertainties in their Stiffnesses", Journal of Sound and Vibration, 42 (4), 477-493, 1975.

[10] Ross, S.M., Introduction to Probability and Statistics for Engineers and Scientists, John Wiley & Sons, New York, NY, 1987.

[11] Soong, T.T., Probabilistic Modeling and Analysis in Science and Engineering, John Wiley & Sons, New York, NY, 1981.